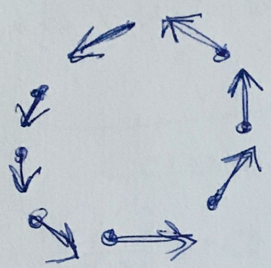
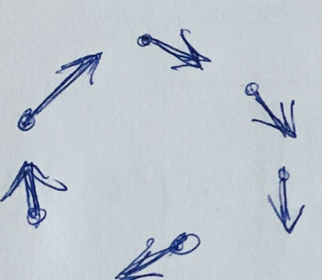
CURL

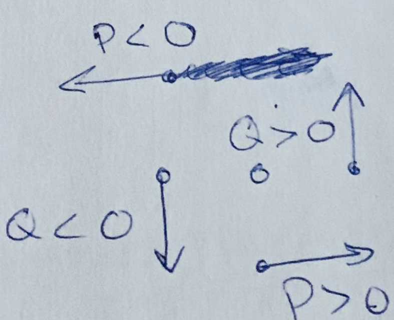


Anti-clockwise Rotation = Positive curl



Clockwise Rotation = Negative curl

No Rotation = zero curl



Positive curl b/c anti-clockwise rotation

curl $\vec{v}(x,y) > 0$ for $\vec{v}(x,y) = \begin{bmatrix} P(x,y) \\ Q(x,y) \end{bmatrix}$

as we go ~~counter clockwise~~ in the positive x-direction, $\underbrace{Q: <0 \to 0 \to >0}_{}$

$$\frac{\partial Q}{\partial x} > 0$$

likewise, we have: $\underbrace{P: >0 \to 0 \to <0}_{}$ as we go the increasing y-direction

$$\frac{\partial P}{\partial y} < 0$$

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$$\therefore, \text{curl}\,\vec{v}(x,y) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \quad \text{for } \vec{v}(x,y) = \begin{bmatrix} P(x,y) \\ Q(x,y) \end{bmatrix}$$

because we saw that

$\frac{\partial Q}{\partial x} > 0$ & $\frac{\partial P}{\partial y} < 0$

corresponds to $\text{curl}\,\vec{v}(x,y) > 0$

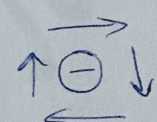
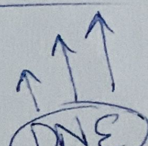
$$\vec{v}(x,y) = \begin{bmatrix} P(x,y) \\ Q(x,y) \end{bmatrix} = \begin{bmatrix} y^3 - 9y \\ x^3 - 9x \end{bmatrix}$$

$$\text{2d curl}\,\vec{v}(x,y) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

$$= (3x^2 - 9) - (3y^2 - 9)$$

$$= 3x^2 - \cancel{9} - 3y^2 + \cancel{9}$$

$$= 3(x^2 - y^2) = 3(x-y)(x+y)$$

<u>COMMON BLUNDER</u> ★ : Curl:  

<u>WRONG</u>!!

By the curl formula,

↑↑↑ has just as much curl/rotation as .

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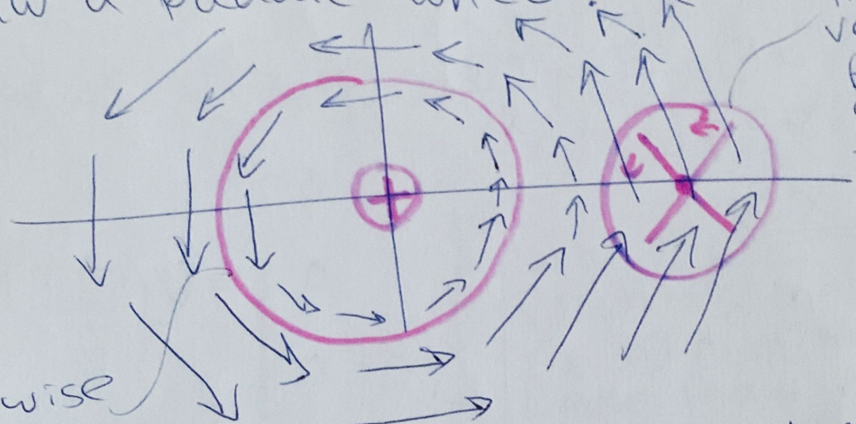
★ TIP: Even if it doesn't look like a vector field has curl or rotation, draw a paddle wheel!



imagine the vectors in the field are exerting a torque on the paddle wheel!...

anti-clockwise rotation = positive curl

let $\vec{v}(x,y) = \begin{bmatrix} P(x,y) \\ Q(x,y) \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$

2d-curl $\vec{v}(x,y) = \dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 1-(-1) = \boxed{2}$

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$\underbrace{\vec{v}(x,y)}_{\text{2 inputs}} = \underbrace{\begin{bmatrix} y^3 - 9y \\ x^3 - 9x \end{bmatrix}}_{\text{2 outputs}}$ } vector valued functions

$$2D\text{-curl } \vec{v}(x,y) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = (3x^2-9)-(3y^2-9)$$
$$= 3x^2 - \cancel{9} - 3y^2 + \cancel{9}$$
$$= 3x^2 - 3y^2$$

$$\text{Curl } \vec{v}(x,y) = \begin{bmatrix} 0 \\ 0 \\ 3x^2 - 3y^2 \end{bmatrix}$$

Q: What's the formula for 3D Curl? We already know that 2D curl is:
$2D \text{ curl } \vec{v}(x,y) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$

$2D \text{ curl } \vec{v}(x,y) = \frac{\partial}{\partial x} Q - \frac{\partial}{\partial y} P$ ← but hey, ... that looks a lot like ... the CROSS PRODUCT!!!

$$2D \text{ curl } \vec{v}(x,y) = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} \times \begin{bmatrix} P \\ Q \end{bmatrix}$$

this definition of the 2D Curl is easily extendable to 3D!!

=> $3D \text{ curl } \vec{v}(x,y,z) = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \times \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$

Curl

$$3D\ \text{curl}(x,y,z) = \underbrace{\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}}_{\nabla} \times \underbrace{\begin{bmatrix} P(x,y,z) \\ Q(x,y,z) \\ R(x,y,z) \end{bmatrix}}_{\vec{v}(x,y,z)} =$$

$$\nabla \times \vec{v}(x,y,z)$$

$$\det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{pmatrix}\hat{i} + \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{pmatrix}\hat{j} + \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{pmatrix}\hat{k}$$

$$= \begin{pmatrix} \left(\frac{\partial}{\partial y}R - \frac{\partial}{\partial z}Q\right)\hat{i} + \\ \left(\frac{\partial}{\partial x}R - \frac{\partial}{\partial z}P\right)\hat{j} + \\ \left(\frac{\partial}{\partial x}Q - \frac{\partial}{\partial y}P\right)\hat{k} \end{pmatrix}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{i} + \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k} =$$

$\text{curl}_{3D}(\vec{v}(x,y,z))$ where $\vec{v}(x,y,z) = \begin{bmatrix} P(x,y,z) \\ Q(x,y,z) \\ R(x,y,z) \end{bmatrix}$

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3D curl = $\nabla \times \vec{V}$ ← So simple! So cool!

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \times \begin{bmatrix} P(x,y,z) \\ Q(x,y,z) \\ R(x,y,z) \end{bmatrix} = \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{pmatrix}$$

$$= \hat{i}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) - \hat{j}\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + \hat{k}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

$$= \begin{bmatrix} \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \\ \frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \\ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \end{bmatrix}$$

$$\vec{V}(x,y,z) = \begin{bmatrix} xy \\ \cos(z) \\ z^2 + y \end{bmatrix} = \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$$

3D curl = $\nabla \times \vec{V}(x,y,z) =$

$$\begin{bmatrix} 1 - (-\sin(z)) \\ 0 - 0 \\ 0 - x \end{bmatrix} = \begin{bmatrix} 1 + \sin(z) \\ 0 \\ -x \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \times \begin{bmatrix} xy \\ \cos(z) \\ z^2 + y \end{bmatrix} =$$