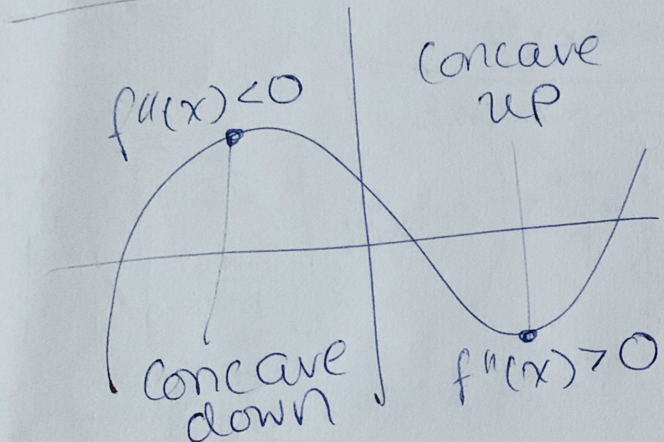
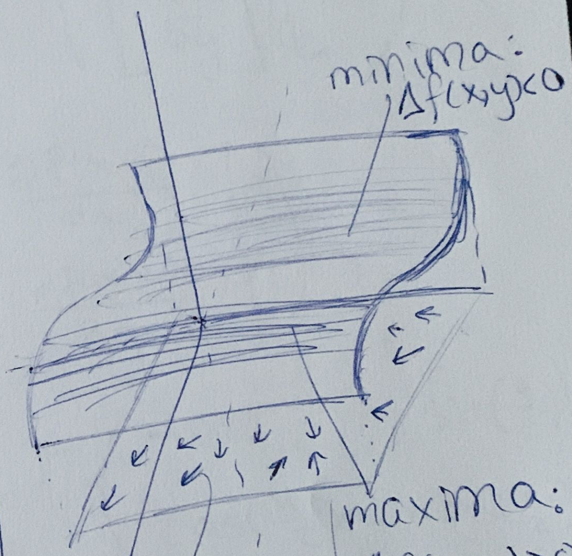
LAPLACIAN

What is the Laplacian? To answer that question, let's look at a 3D surface.

2D



3D



gradient is ∇f and produces a vector field

$$\Delta f(x,y) = \underbrace{\nabla \cdot \nabla f}_{\text{div(grad} f)} \quad \nabla \circ \nabla f$$

scalar function

gradient of f returns a vector field

divergence of a vector field is a SCALAR.

LAPLACIAN

$f(x,y) = 3 + \cos\left(\frac{x}{2}\right)\left(\sin\left(\frac{y}{2}\right)\right)$

compute the $\underbrace{\text{laplacian.}}$ $\rightarrow$ But how?

$$\Delta f(x,y) = \underbrace{\nabla \circ \underbrace{\nabla f(x,y)}_{\text{gradient}}}_{\text{divergence}} = \nabla \circ \left(\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} f \right)$$

$$\nabla \circ \left(\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} f(x,y) \right) = \nabla \circ \left(\begin{bmatrix} \frac{\partial f(x,y)}{\partial x} \\ \frac{\partial f(x,y)}{\partial y} \end{bmatrix} \right)$$

$$= \nabla \circ \begin{pmatrix} \frac{\partial}{\partial x} \cos\left(\frac{x}{2}\right)\sin\left(\frac{y}{2}\right) \\ \frac{\partial}{\partial y} \cos\left(\frac{x}{2}\right)\sin\left(\frac{y}{2}\right) \end{pmatrix} = \nabla \circ \begin{pmatrix} -\sin\left(\frac{x}{2}\right) \cdot \frac{1}{2} \sin\left(\frac{y}{2}\right) \\ \cos\left(\frac{y}{2}\right) \circ \frac{1}{2} \cdot \cos\left(\frac{x}{2}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} \circ \begin{pmatrix} -\frac{1}{2} \sin\left(\frac{x}{2}\right) \sin\left(\frac{y}{2}\right) \\ \frac{1}{2} \cos\left(\frac{x}{2}\right) \cos\left(\frac{y}{2}\right) \end{pmatrix} = \frac{\partial}{\partial x}\left(-\sin\left(\frac{x}{2}\right) \cdot \frac{1}{2} \cdot \sin\left(\frac{y}{2}\right)\right) + \frac{\partial}{\partial y}\left(\frac{1}{2}\cos\left(\frac{x}{2}\right)\cos\left(\frac{y}{2}\right)\right)$$

$$= \frac{\partial}{\partial x}\left(-\frac{1}{2}\sin\left(\frac{x}{2}\right)\sin\left(\frac{y}{2}\right)\right) + \frac{\partial}{\partial y}\left(\frac{1}{2}\cos\left(\frac{x}{2}\right)\cos\left(\frac{y}{2}\right)\right)$$

$$= -\frac{1}{2}\sin\left(\frac{y}{2}\right)\frac{\partial}{\partial x}\left(\sin\left(\frac{x}{2}\right)\right) + \frac{1}{2}\cos\left(\frac{x}{2}\right)\frac{\partial}{\partial y}\left(\cos\left(\frac{y}{2}\right)\right)$$

$$= -\frac{1}{2}\sin\left(\frac{y}{2}\right)\frac{1}{2}\cos\left(\frac{x}{2}\right) + \frac{1}{2}\cos\left(\frac{x}{2}\right)\left(-\sin\left(\frac{y}{2}\right)\right)\left(\frac{1}{2}\right)$$

LAPLACIAN

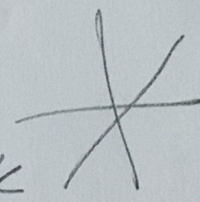
$$\underbrace{\Delta f(x,y) = \nabla \cdot \nabla f(x,y) \equiv 0}$$

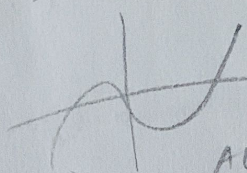
criteria for a <u>harmonic</u> <u>function</u>

$f''(x) = 0$

$f'(x) = c$

$f(x) = cx + k$

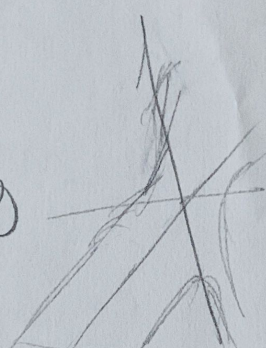


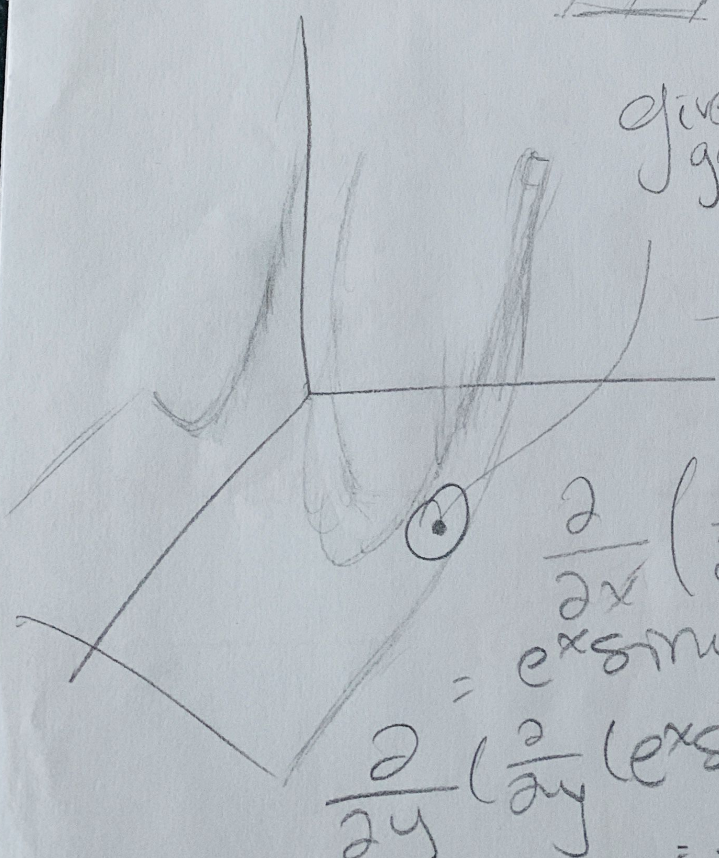


no max
no min
no curves
ALLOWED!

$f(x,y) = e^x \sin(y)$

$$\nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \stackrel{?}{=} 0$$





given any point on the graph, all the outputs average to the center point

think of stability, etc, ...

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}(e^x \sin y)\right) = \frac{\partial}{\partial x}(e^x \sin y)$$

$$= e^x \sin y$$

$$\frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}(e^x \sin(y))\right) = \frac{\partial}{\partial y}(\cos(y) e^x)$$

$$= -\sin(y) e^x \Rightarrow \begin{array}{r} e^x \sin y \\ - e^x \sin y \\ \hline 0 \end{array}$$

LAPLACIAN

$f(x_1, x_2, \dots, x_N)$ $\quad \Delta f = \nabla \circ \nabla f$

$$\begin{bmatrix} \partial/\partial x_1 \\ \partial/\partial x_2 \\ \partial/\partial x_3 \\ \vdots \\ \partial/\partial x_n \end{bmatrix} \circ \left(\begin{bmatrix} \partial/\partial x_1 \\ \partial/\partial x_2 \\ \partial/\partial x_3 \\ \vdots \\ \partial/\partial x_n \end{bmatrix} \cancel{\begin{bmatrix} P_1(x_1, x_2, \dots) \\ P_2(x_1, x_2, \dots) \\ P_3(x_1, x_2, \dots) \\ \vdots \\ P_n(x_1, x_2, \dots) \end{bmatrix}} f \right) =$$

$$\begin{bmatrix} \partial/\partial x_1 \\ \partial/\partial x_2 \\ \vdots \\ \partial/\partial x_n \end{bmatrix} \circ \begin{bmatrix} \partial P_1(x_1, x_2, \dots)/\partial x_1 \\ \partial P_2(x_1, x_2, \dots)/\partial x_2 \\ \vdots \\ \partial P_n(x_1, x_2, \dots)/\partial x_n \end{bmatrix} = \frac{\partial}{\partial x_1}\left(\frac{\partial \overset{f}{\cancel{P_1(x_1, x_2, \dots)}}}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{\partial \overset{f}{\cancel{P_2(x_1, x_2, \dots)}}}{\partial x_2}\right) + \dots + \frac{\partial}{\partial x_n}\left(\frac{\partial \overset{f}{\cancel{P_n(x_1, x_2, \dots)}}}{\partial x_n}\right)$$

$$= \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \dots + \frac{\partial^2 f}{\partial x_n^2} = \sum_{k=1}^{n} \frac{\partial^2 f}{\partial x_k^2} \Rightarrow \sum_{k=1}^{n} \frac{\partial^2 f}{\partial x_k^2}$$

$$\boxed{\text{Laplacian} = \underbrace{\nabla \circ \underbrace{\nabla f}_{\text{grad}}}_{\text{div}} = \sum_{k=1}^{n} \frac{\partial^2 f}{\partial x_k^2}}$$