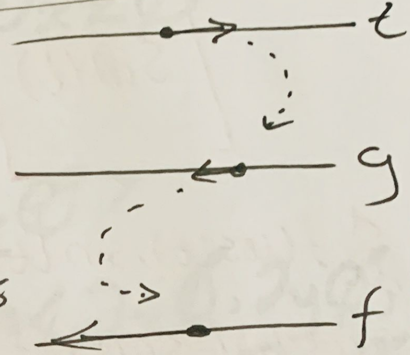
MULTIVARIABLE CHAIN RULE:

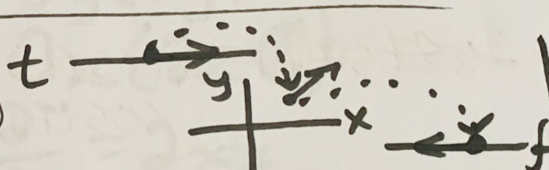
$$\frac{d}{dt} f(x(t), y(t)) = \underbrace{\frac{\partial f}{\partial x} \cdot \frac{dx}{dt}} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

$$\frac{d}{dx}\left(f(g(t))\right) = \underbrace{\frac{df}{dg}}_{\text{how a tiny change in g influences f(g(t))}} \cdot \underbrace{\frac{dg}{dt}}_{\text{how much a tiny change in t influences g(t)}}$$



SINGLE VARIABLE

$$\vec{v}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \qquad f(\vec{v}(t))$$



$$f(x(t), y(t)) = f(\vec{v}(t))$$

$$\frac{d}{dt} f(\vec{v}(t)) = \frac{\partial f}{\partial x}(\vec{v}(t)) \frac{dx}{dt} + \frac{\partial f}{\partial y}(\vec{v}(t)) \frac{dy}{dt}$$

$$\nabla_{\vec{v}} f = \frac{\partial f}{\partial v} = \frac{d}{dx} f(\vec{v}(t))$$

$$\begin{bmatrix} \partial/\partial x\, f(\vec{v}(t)) \\ \partial/\partial y\, f(\vec{v}(t)) \end{bmatrix} \circ \begin{bmatrix} dx/dt \\ dy/dt \end{bmatrix}$$

$$\frac{d}{dx} f(\vec{v}(t)) = \nabla f(\vec{v}(t)) \cdot \vec{v}\,'(t)$$

MULTI VARIABLE

MULTIVARIABLE CHAIN RULE

$$\frac{d}{dx}\left(f(g(x))\right) = f'(g(x))\, g'(x)$$

$$\frac{d}{dx}\left(\underline{\sin(\underline{x^2})}\right) = \cos(x^2) \cdot 2x$$

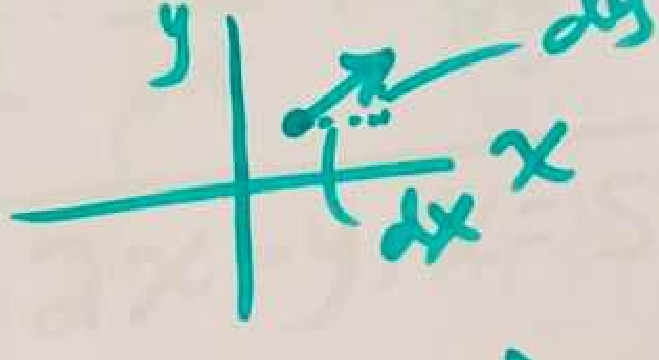
$$f(x,y) = f(x(t), y(t))$$

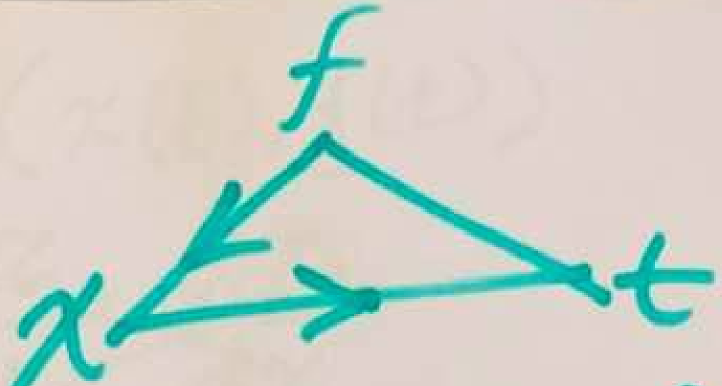
$$\cancel{\frac{df}{dx}} \quad \cancel{\frac{\partial f}{\partial x}} \quad \frac{\partial f}{\partial t}$$

$$\frac{d}{dt}\left(f(x(t),y(t))\right) =$$

$$= \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

$$= \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$





$$dt \rightarrow dx \rightarrow \partial f \}$$

$$dt \rightarrow dy \rightarrow df \}$$

$$\frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

$z = 2x - y, x = \sin t,$

$y = 3t$

$$Z(x(t), y(t))$$

$$\cancel{\frac{dz}{dt} = \frac{dz}{dx}}$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$\frac{\partial}{\partial x}(2x - y) = 2$$

$$\frac{dx}{dt} = \frac{d}{dt}(\sin t) = \cos t$$

$$\frac{\partial}{\partial y}(2x - y) = -1 \quad \Bigg| \quad \frac{dy}{dt} = 3$$

$$\frac{dz}{dt} = (2)(\cos t) - (1)(3)$$

$$= \boxed{2\cos t - 3} \checkmark$$

$z = x\sin(y)$, $x = e^t$,

$y = 3t$

$$\frac{dz}{dt} = \cancel{\frac{dx}{dt}}\ \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$\frac{\partial z}{dx} = \frac{\partial}{\partial x}\left(x\sin(y)\right) = \sin(y)$$

$$\frac{dx}{dt} = \frac{d}{dt}(e^t) = e^t$$

$$\frac{\partial z}{\partial y} = \cos y \; x$$

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$$\frac{dy}{dt} = 3 \quad \Big| \; e^t \sin(3t) + 3e^t \cos(3t)$$

$$\frac{dx}{dt} = \sin(y)e^t + x\cos(y) \cdot 3$$

$$= e^t \sin(y) + 3x\cos(y)$$

$$z = xy + y^2, \; x = t^2, \; y = t+1$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$\frac{\partial z}{\partial x} = y \quad \Big| \quad \frac{dx}{dt} = 2t \quad \Big| \quad \frac{\partial z}{\partial y} = x + 2y$$

$$\frac{dy}{dt} = 1$$

$$\frac{dz}{dt} = (y)(2t) + (x+2y)$$

$$= 2yt + x + 2y$$

$$= 2(t+1)t + t^2 + 2(t+1)$$

$$= (2t+2)t + t^2 + 2t + 2$$

$$= 2t^2 + 2t + t^2 + 2t + 2$$

$$= 2t^2 + 4t + t^2 + 2$$

$$= 2t^2 + 4t$$

$$= \boxed{3t^2 + 4t + 2}$$

$$z = \ln\left(\frac{x^2}{y}\right), x = e^t, y = t^2$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$\frac{\partial}{\partial x}\left(\ln\left(\frac{x^2}{y}\right)\right) = \left(\frac{1}{x^2}\right) 2x = \frac{2}{x}$$

$$\ln\left(\frac{x^2}{y}\right) = \ln x^2 - \ln y \rightarrow 0$$

$$\frac{\partial z}{\partial x} = \frac{2}{x} = \frac{2}{e^t} \qquad \frac{dy}{dt} = 2t$$

$$\frac{dx}{dt} = e^t \qquad \frac{\partial z}{\partial y} = \cancel{\ln x^2} - \ln y \rightarrow 0$$

$$= -\frac{1}{y} = -\frac{1}{t^2}$$

$$\frac{dz}{dt} = \frac{2}{\cancel{e^t}} \cdot \cancel{e^t} + \frac{1}{t^2} \cdot 2t$$

$$= 2 - \frac{2}{t}$$

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$\cancel{z}(x(t), y(t)) =$
w

$w = x^2 + y^2 + \underline{\underline{2x^2}}$

$x = t + 1,\ y = \cos t,$
$z = \sin t$

$\cancel{\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt}}$

$\cancel{\frac{\partial w}{\partial x} = \frac{\partial}{}}$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dt}$$

$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial t}(x^2+y^2+2z^2)$$

$$= 2x = 2(t+1) = 2t+2$$

$$\frac{dx}{dt} = 1 \quad \Big| \quad \frac{\partial w}{\partial y} = 2y = 2\cos t$$

$$\frac{dy}{dt} = \frac{d}{dt}(\cos t) = -\sin t$$

$$\frac{\partial w}{\partial z} = \frac{\partial}{\partial z}(x^2+y^2+2z^2) = 4z$$

$$= 4\sin(t)$$

$$\frac{dz}{dt} = \cos t$$

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$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dt}$$

$$= (2t+2) + (2\cos(t))(-\sin(t)) + 4\sin(t)\cos(t)$$

$$= 2t+2 - 2\sin(t)\cos(t) + 4\sin(t)\cos(t)$$

$$\sin(2x) = 2\sin x \cos x$$

$$= 2t+2 - \sin(2t) + 2\sin(2t)$$

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$$\frac{dw}{dt} = 2t + 2 + 2\sin(2t)$$

$$G(u, v) = f(3u - v, u^2 + v)$$

$$\left.\frac{\partial g}{\partial u}\right|_{(u,v)=(2,-1)}$$

$$\left.\frac{\partial g}{\partial v}\right|_{(u,v)=(2,-1)}$$

(x,y)	f	g	f_x	f_y
$(2,-1)$	6	-7	1	9
$(7,3)$	4	2	-3	5

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$$\frac{\partial g}{\partial u}\Big|_{(u,v)=(2,-1)} = g_u(2,-1)$$

$$g(u,v) = f(3u-v,\, u^2+v)$$

$$\frac{\partial g(u,v)}{\partial u} = \frac{\partial f(u,v)}{\partial u}$$

$$\cancel{\frac{\partial f}{\partial u}} = f(3u-v,\, u^2+v)$$

$$f(x,y);\ \begin{aligned} x &= 3u - v \\ y &= u^2+v \end{aligned}$$

$$f_u(2,-1) \Rightarrow f(x(u,v),\, y(u,v))$$

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$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} \cdot \frac{dx}{du} + \frac{\partial f}{\partial y} \cdot \frac{dy}{du}$$

$$\frac{\partial}{\partial u}(3u - v) = 3$$

$$\frac{\partial}{\partial u}(u^2 + v) = 2u$$

$$\frac{\partial f}{\partial u} = 3 + 2u = 3 + 2(2)$$
$$= 7$$

$\sqrt{14}$

Define $g(u,v) = f(3u-v, u^2+v)$. Find $\left.\frac{\partial g}{\partial u}\right|_{2,-1}$ & $\left.\frac{\partial g}{\partial v}\right|_{2,-1}$.

(x,y)	f	g	f_x	f_y
$(2,-1)$	6	-7	1	9
$(7,3)$	4	2	-3	5

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

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$$\frac{\partial g}{\partial u} = \overset{f_x}{\frac{\partial f}{\partial x}} \cdot \frac{dx}{du} + \overset{f_y}{\frac{\partial f}{\partial y}} \cdot \frac{dy}{du}$$

$$\frac{\partial f}{\partial x}\ ;\ x = 3u - v,\ y = u^2 + v$$

$$\frac{dx}{du} = 3 \quad \Big| \quad \frac{dy}{du} = 2u = 4$$

$$\frac{\partial g}{\partial u} = f_x \frac{dx}{du} + f_y \frac{dy}{du}$$

$$= f_x \boxed{\frac{d}{du}(3u - v)} + f_y \boxed{\frac{d}{du}(u^2 + v)}$$

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$= f_x(3) + f_y(4) = 3 + 36$

$= \boxed{39}$

$= f_x(3) + f_y(2u)$

$= 3f_x + 2u f_y$

$(u, v) \neq (x, y)$

$(x, y) = (3u - v, u^2 + v)$

$(x, y) = (3 \cdot 2 - (-1), (2)^2 + (-1))$

$\rightarrow = (7, 3) = (x, y)$

$= 3f_x + 4f_y = 3(-3) + 4(5)$

$= \boxed{11}$

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$$\left.\frac{\partial g}{\partial v}\right|_{2,-1} =? \frac{\partial f}{\partial x}\cdot\frac{dx}{dv} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dv}$$

$$f_x \frac{dx}{dv} + f_y \frac{dy}{dv}$$

$$f_x \frac{d}{dv}(3u-v) + f_y \frac{d}{dv}(u^2+v)$$

$$f_x(-1) + f_y(1) = f_y - f_x$$

$$(x,y) = (3u-v, u^2+v)$$

$$\cancel{(-3} \; 5-(-3) = \boxed{8}$$

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