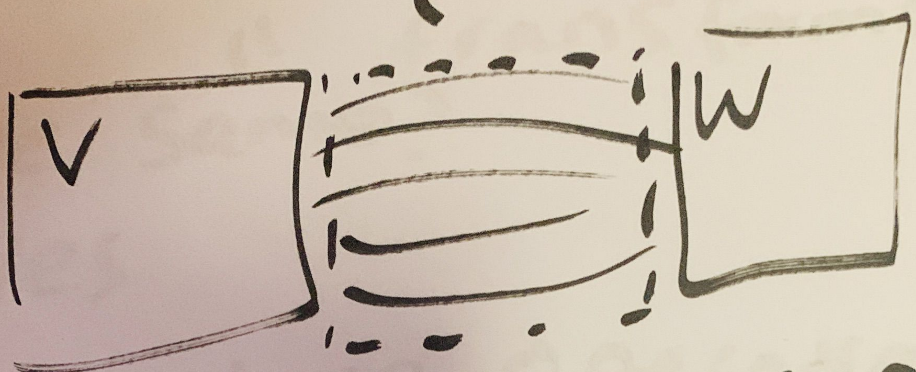
Hom(V,K)



__Def__: let V, W be vec. spaces over field F. The __collection__ of all linear maps from V to W is denoted Hom(V,K).

~~t~~ homomorphisms

Is the set of $\{\mathbb{R}\}$ a field?
yes. A field is any algebiaic structure where $+, -, \times, \div$ are defined

Are linear maps and
" transformations
the same?

yes.

Structure-Operations-
Fields-Groups-space

(*) structures
(operations)

Fields — Vector Spaces

Fields: $\{\mathbb{R}\}$ $\{\mathbb{Q}\}$

Vector Spaces: $\mathbb{R}$ $\mathbb{C}$

SETS

* CAN WE IMPOSE A STRUCTURE ON Hom(V,W)?

Yes! It has at least 2 operations $<$ + ; $\times$ by scalar

Thm: Hom(V,W) is a vector space over F wrt the addition of linear maps & scalar mult.

to be a vector space... this is **NOT** a subspace!

> commutativity: is $S+T = T+S$

> distributive:

>Commutativity:

Is $s+t=t+s$?

>Distributivity:

Is $\alpha(\beta+\gamma)=\alpha\beta+\alpha\gamma$

>Zero Element:

- What's the 0 element of $Hom(V,k)$?
- What <u>are</u> the elements of $Hom(V,k)$?

  >Linear maps!

  >0 map!!! That sends everyone to 0!!!

Given a linear map

$T: V \to W$

choose a basis for T that **represents** that linear map wrt that basis and you have a **MATRIX**! ~~that~~

matrix → linear map

basis

In fact....
matrix = LINEAR MAPs

The space of Hom(V,W)
is *ISOMORPHIC* to
matrices.

if:

$T, S: V \to W$
$\in \mathbb{R}^a \quad \in \mathbb{R}^b$
where $a \neq b$

and E is a basis for V
& F " " " W

$$[T+S]_E^F = [T]_E^F + [S]_E^F$$

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$[T+S]_E^F = [T]_E^F + [S]_E^F$

$[\alpha T]_E^F = \alpha [T]_E^F$

$K: T \to [T]_E^F$

a linear map!

The map

$\varphi : \mathrm{Hom}(V,W) \to M_{\underline{m \times n}}(F)$

range

$\dim(V) = m \times m$

$\dim(W) = n \times n$

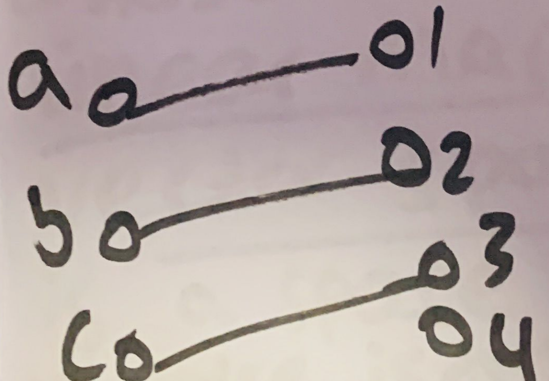
$$\underbrace{\varphi(T) = [T]_E^F}_{\text{is a linear map!}}$$

IF I can show:

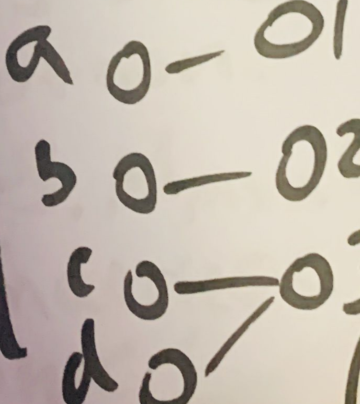
φ is 1-1

& onto

1-1 not onto:



onto not 1-1:



We'll show that
φ is 1-1 & onto
and conclude
that $Hom(V,W) \cong M_{m \times n}(F)$

Zero element

Hom(V,W) is a vector space
whose elements are
linear maps btwn.
vector spaces. $M_{m \times n}(F)$
is a space of matrices.
They're isomorphic!

Thus, $\exists \varphi$ s.t.

$\varphi : \mathrm{Hom}(V,W) \to M_{m \times n}(F)$

$$\varphi(T) = [T]_E^F$$

Or equivalently;

$$\dim(\mathrm{Hom}(V,W)) = (\dim(V))(\dim(W))$$

V 4

e_μ $\mathbb{R}$

Λ $\rightarrow$ $\mathbb{R}$

$\langle \Lambda, e_\mu \rangle$

Λ_μ

$V^*: V \to \mathbb{R}$

DUAL SPACES!!

$$\boxed{\begin{matrix} V & 4 \\ \ell m & \mathbb{R} \end{matrix}} \xrightarrow{\Lambda} \boxed{\mathbb{R}}$$

$\left\{ \text{all possible maps} \right\} = V^*$ **DUAL SPACE**

$$\langle \Lambda + Z, v \rangle = \langle \Lambda, v \rangle + \langle Z, v \rangle$$

$$\langle a\Lambda, v \rangle = a \langle \Lambda, v \rangle$$

$\boxed{v}$

ORIGINAL

$V \quad 4$

$e_\mu \quad \mathbb{R}$

DUAL SPACE

$V^* \quad 4$

$e^\mu \quad \mathbb{R}$

e_μ

(basis vector from normal)

e^μ

(basis vector from dual space)

linear maps!

dual basis

$\langle e^\mu, e_\nu \rangle = \delta^\mu_\nu$

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