LINE $\oint$ (VECTOR FIELDS!)

What?

gravitational field
magnetic fields

$$\underbrace{\vec{F}(x,y) = P(x,y)\hat{i} + Q(x,y)\hat{j}}$$

$$F: \mathbb{R}^2 \to \mathbb{R}^2$$

VS

Scalar field:

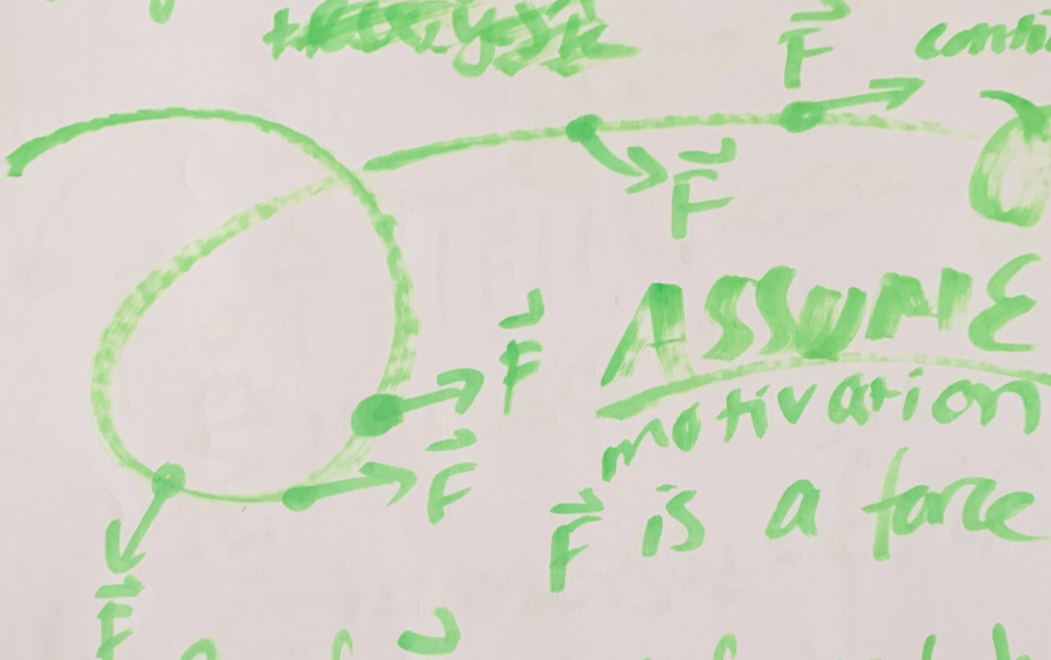
$$S: \mathbb{R}^2 \to \mathbb{R}^1$$

ex: $\vec{\nabla} f(x,y) = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$

the gradient is a VECTOR FIELD!!!!!

$\vec{F}(x,y) = P(x,y)\hat{\imath} + Q(x,y)\hat{\jmath} \in \underbrace{C^1}_{\text{continuous}}$

~~$+R(x,y)\hat{k}$~~



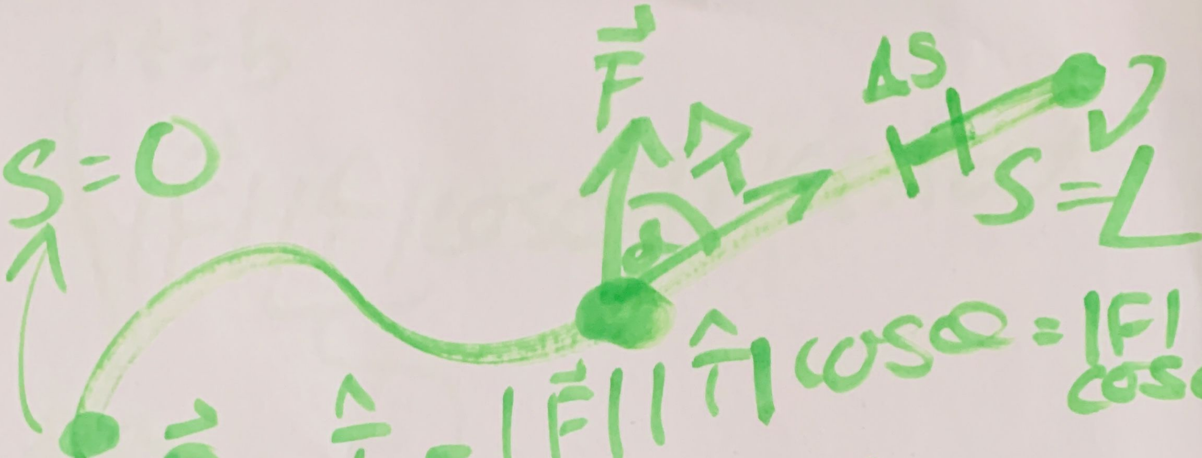
ASSUME:

<u>motivation</u>

$\vec{F}$ is a force.

<u>Key Q</u>: If $\vec{F}$ is a force, what is the work done by $\vec{F}$ along γ?

<u>Recall</u>: $W = Fd$ ...if $\vec{F}$ is constant



$$\vec{F} \circ \hat{T} = |\vec{F}||\hat{T}|\cos\theta = |F| \cos\theta$$

$$\sum_{i=1}^{n} |F|\cos\theta \, \Delta S_i$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} |F|\cos\theta \, \Delta S_i = \oint_{s=0}^{s=L} \vec{F}\circ\hat{T} \, dS$$

$$\oint \vec{F}\circ\hat{T}\,|r'(t)|\,dt \Leftarrow \oint_a^b \vec{F}(x(t),y(t)) \cdot \hat{T}(x(t),y(t)) \sqrt{dx^2+dy^2}$$

$$dt\,|r'(t)| \Leftarrow \frac{dt}{dt}\sqrt{dx^2+dy^2} = dt\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$$

$$\int_{t=a}^{t=b} |\vec{F}||\hat{T}|\cos\theta\,|r'(t)|\,dt$$

$$\int_{t=a}^{t=b} \cancel{|\vec{F}|}\,\vec{F}\circ\frac{r'(t)}{\cancel{|r'(t)|}}\,\cancel{|r'(t)|}\,dt$$

$$\int_{t=a}^{t=b} \vec{F}\circ\vec{r}\,'(t)\,dt$$

$$\boxed{\textbf{Def.:}\ \int_{\gamma}\vec{F}\cdot d\vec{r}=\int_{a}^{b}\vec{F}\circ|r'(t)|\,dt}$$

$$\int_a^b \vec{F}(x(t), y(t)) \cdot \left(\frac{dr}{dt} dt \right)$$

$$= \int_a^b \vec{F} \cdot d\vec{r}$$

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$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$$

also written...

$$\int_\gamma \vec{F} \cdot d\vec{r} = \int_a^b P(x(t), y(t))\, x'(t) + Q(x(t), y(t))\, y'(t) \, dt$$

$$\left. \begin{array}{l} \vec{F} = (P, Q) \\ \vec{r}\,' = (x', y') \end{array} \right\}$$

$$\int_a^b \left(P\frac{dx}{dt} + Q\frac{dy}{dt}\right)dt = \int_a^b P\,dx + Q\,dy$$

Ex: $\vec{F}(x,y,z) = x\hat{i} + y\hat{j} + z\hat{k}$

$P(x,y,z)$ $Q(x,y,z)$ $R(x,y,z)$

γ = helix:

$\vec{r}(t) = (\cos t, \sin t, t)$

$0 \le t \le 2\pi$

$r'(t) = (-\sin t, \cos t, 1)$

how do you paramatrize $\vec{F}$???

6

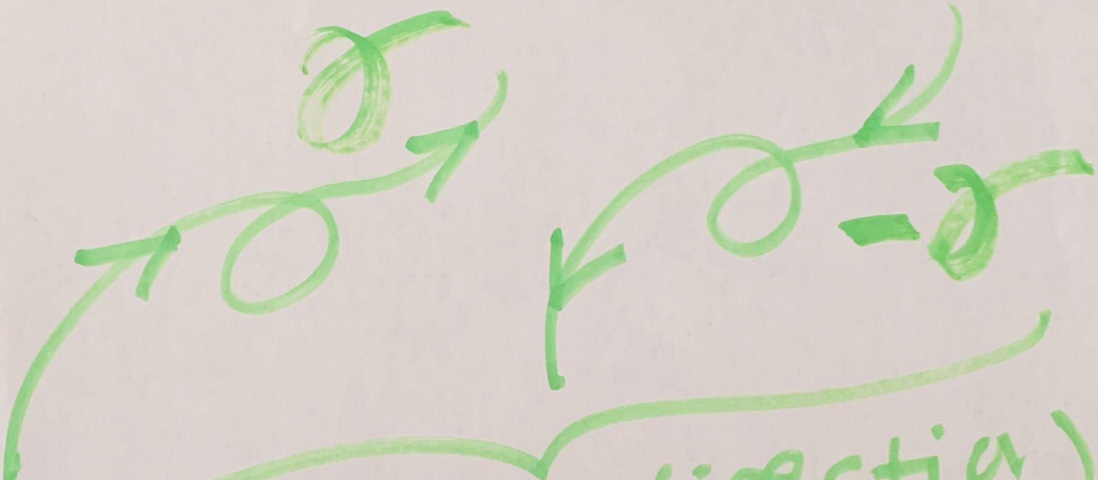
$$\int_\gamma \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \underbrace{(\cos t, \sin t, t)}_{\substack{F(x(t),\, y(t),\, z(t)) \\ P(t),\, Q(t),\, R(t)}} \cdot (-\sin t, \cos t, 1)\, dt$$

$$\int_0^{2\pi} -\sin t \cos t + \sin t \cos t + t \; dt = \frac{t^2}{2}\Big|_0^{2\pi} = \underline{\underline{2\pi^2}}$$

<u>remarks</u>:

1. $\int \vec{F} \cdot d\vec{r}$ doesn't depend on parametrization

2. It <u>does</u> depend on the direction (orientation):



Switching direction <u>negates</u> work.

$$\int_{-\gamma} \vec{F}\cdot d\vec{r} = -\int_{\gamma} \vec{F}\cdot d\vec{r}$$