

A Bridge from Maxwell to Einstein

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January 15, 2021

Abstract

There exists a gap in the literature of Physics Education: the bridge between Maxwell's Equations and Einstein's Theory of Special Relativity remains obscure in many Undergraduate Electromagnetics textbooks even today. Furthermore, in the few textbooks that do cover it, many have failed to take advantage of applying computational programming to better highlight the relationship between Maxwell and Einstein. Although much of the mathematics in this paper has been tackled previously in the literature, a visual demonstration of these concepts via programming suitable for undergraduates has never been applied, thus leaving a serious gap in the literature. The author seeks to resolve this gap by illustrating three fundamental principles that make apparent the bridge from Maxwell to Einstein via computational programming in Python: (1) Origin of the Lorentz Gamma Factor from Maxwell's Equations, (2) Time Dilation from a Classic Electromagnetics Scenario, and (3) Derivation of the Displacement Current via the Biot-Savart Law and Special Relativity. The visual application of these ideas pose many pedagogical opportunities and extensions in the classroom, which are also discussed. All computational visualizations are available [here](#) Should the above link fail, a fallback option is provided [here](#)

1 Introduction

It's been 115 years since the inception of Einstein's Special Theory of Relativity [1] (STR) and another 155 years since Maxwell's Unification of the Laws of Electromagnetism [2]. And yet, the relationship between these two fundamental laws of the universe remain obscure to many Physics and Engineering students to this day. The problem is two-fold: a) Few undergraduate electromagnetics textbooks highlight the inseparable relationship between Maxwell and Einstein's Theories of EM and STR [3], and b) the few that do have failed to take advantage of the digital revolution [4] [5]. Instead, many supply antiquated diagrams that fail to scaffold intuition, as opposed to actual programs that reinforce both a student's theoretical understanding and intuitive grasp [6]. This paper seeks to resolve this gap in the literature of Physics Education by illustrating three fundamental elements that highlight the bridge between Maxwell and Einstein, all via computational programming: (1) Origin of the Lorentz Gamma Factor from Maxwell's Equations, (2) Time Dilation from a Classic Electromagnetics Scenario, and (3) Derivation of the Displacement Current via the Biot-Savart Law.

2 Literature Review

It is a well-documented fact that many students struggle not only with the bridge between Maxwell's Equations and Einstein's Special Theory of Relativity, but with Maxwell's Equations themselves! [10, 11] The fact of the matter is that there exists an inherent gap in Physics Education – despite the introduction of Visual Computing some 20 years ago and extensive documentation and resources on the subject [22], few if any classes have adapted the teaching of these critical ideas to this new digital age [23] [24]. Instead, students are left trying to haphazardly conjecture what Gaussian Surface or Amperian Loop to draw given the symmetry of a surface, when the fundamental gap lies in the execution of the material, not its difficulty [12]. Indeed, we must make EM and STR hands-on, much as Common Core has done to the Algebra curriculum [16]. The best approach to making the ideas of EM and STR tangible without the hassle of experimentation (i.e., collecting data, calculating error, calibrating equipment, etc.) is computational programming. Indeed, the advent of languages and libraries such as Python, Plot.ly, GlowScript, and WebGL has made the transition accessible, if not downright trivial [21].

To demonstrate the utility of using Computational Programming to highlight the bridge between Maxwell's Equations and the Special Theory of Relativity, this paper tackles three elements of that bridge that have traditionally served as barriers for undergraduate students:

1. Lorentz Gamma Factor from Maxwell's Equations
2. Time Dilation from Electromagnetic Scenario
3. Displacement Current from Biot-Savart Law and Special Relativity

Each of these principles are associated with a host of student misconceptions, many of which have been extensively documented – conundrums such as whether the Displacement Current is a real current [20] and produces a magnetic field or how an electric field could be analogous to a magnetic field depending on the reference frame of an observer. All these misconceptions and confusions may be addressed using the computational visualizations outlined within the paper. As such, the student will emerge with a much greater intuitive grasp of the bridge between Maxwell's Equations and Einstein's Theory of Special Relativity. We begin our exploration of this bridge with the derivation of the Lorentz Transformation from Maxwell's Equations.

3 γ from Maxwell's Equations

Deriving the Lorentz Transformations provides one a special insight into Einstein's Special Theory of Relativity. In fact, the famous Gamma factor that gives rise to the Lorentz Transformations is contained within Maxwell's Equations. As such, deriving γ from Maxwell's Equations is the foundation of the bridge between Maxwell and Einstein, and it is the principle we explore first.

The motivating basis for the Lorentz Transformations is that the Galilean Transformations fails to maintain the invariance of Maxwell's Equations. Many students grapple with

the inconsistency of the Galilean Transformations, but to no avail. Indeed, many struggle with the basic Galilean Principle of Relativity itself [29]. The resolution to this problem is not supplying students with yet another geometric representation of the Galilean Transforms, but providing an actual interactive visualization of it. By interacting with the abstract principle of Relativity – even at the basic level of Galilean Transforms – students will begin scaffolding a much stronger intuition for the greater picture of relativity.

The derivation of the Lorentz Gamma Factor from Maxwell’s Equations begins with Maxwell’s Equations themselves, from which the Wave Equation will be derived. The Galilean Transformations are then introduced via an interactive, computational simulation that enables the user to provide a velocity for two observers in relative motion witnessing an event E . Given that intuitive basis for the Galilean Transformation, we test whether the Wave Equation is invariant under the Galilean Transformation. Once this fails, we test the Fitzgerald Transformations, and finally the Lorentz Transformations – a full derivation is supplied for all aforementioned transformations. The final result is the Lorentz Gamma Factor from Maxwell’s Equations, a satisfactory result that will provide students a theoretical and intuitive understanding.

To derive the γ from Maxwell’s Equations, we begin with a statement of Maxwell’s 4 Equations of Electromagnetism in their integral form.

$$\begin{aligned}
 \Phi_E &= \oint E \cdot dA = \frac{q_{enc}}{\mathcal{E}_0} \\
 \Phi_B &= \oint B \cdot dA = 0 \\
 \oint E \cdot dS &= -\frac{d\Phi_B}{dt} \\
 \oint B \cdot dS &= \mu_0 \mathcal{E}_0 \frac{d\Phi_E}{dt} + \mu_0 J
 \end{aligned} \tag{1}$$

These four equations encompass all of electromagnetic theory. And indeed, they serve as the bridge to Special Relativity. We convert the integral form of the equations to their differential form via Green’s and Stokes Theorem. For brevity, the differential form of Maxwell’s Equations are given as

$$\begin{aligned}
 \nabla \cdot E &= \frac{\rho}{\mathcal{E}_0} \\
 \nabla \cdot B &= 0 \\
 \nabla \times E &= -\frac{\partial B}{\partial t} \\
 \nabla \times B &= \mu_0 \mathcal{E}_0 \frac{\partial E}{\partial t} + \mu_0 J
 \end{aligned} \tag{2}$$

With the differential form of Maxwell’s Equations, we now proceed to deriving the Wave Equation, which describes the transverse nature of Electromagnetic Waves. After deriving the Wave Equation, we seek to visualize the Galilean Transformations (GT) and apply it to the Wave Equation to test whether Maxwell’s Equations hold invariant under the GT. The Fitzgerald and Lorentz Transformations are then applied to the Wave Equation in the

hopes of maintaining its invariance. Finally, the utility and applications of the Lorentz Transformations are briefly investigated through Length Contraction, Time Dilation, and the constancy of the speed of light.

3.1 The Wave Equation

From hereon, we can derive the second-order partial differential that is the Wave Equation.

$$\nabla \times B = \mu_0 \mathcal{E}_0 \frac{\partial E}{\partial t} + \mu_0 J \quad (3)$$

We now convert to light units where $c = 1$. Since $\mu_0 \mathcal{E}_0 = \frac{1}{c^2}$, we thus have that $\nabla \times B = \frac{\partial E}{\partial t} + \mu_0 J$. We begin by considering Ampere's Law in a vacuum (i.e., $J, \rho = 0$)

$$\begin{aligned} \nabla \times B &= \frac{\partial E}{\partial t} + \mu_0 J \\ \nabla \times B &= E_t \\ \frac{\partial}{\partial t}(\nabla \times B) &= E_{tt} \\ \nabla \times B_t &= E_{tt} \\ \nabla \times \frac{\partial B}{\partial t} &= E_{tt} \\ -\nabla \times (\nabla \times E) &= E_{tt} \\ -\nabla(\nabla \cdot E) + \nabla^2 E &= E_{tt} \end{aligned} \quad (4)$$

And thus we have the wave equation as

$$E_{tt} = \nabla^2 E \quad (5)$$

We now test whether the Wave Equation is invariant under the Galilean Transformation. First and foremost, however, we must understand the Galilean Transformations intuitively. To do so, we encounter our first computational programming exercise: creating an interactive visualization of the Galilean Transformations.

3.2 Visualizing the Galilean Transformation

Our goal is to construct a visualization of the Galilean Transformation suitable for undergraduates, that enables them to grasp the key idea of relativity – all observers in relative motion share the same time and conjecture identical physical laws for the universe, but witness events at a different location. Our visualization will be based on the following scenario:

Rob (R) and Greg (G) are in relative motion to each other. R is moving down G's ζ axis. R has coordinates (t, x, y, z) and G has coordinates (τ, ξ, η, ζ) . They synchronize their clocks at an Event O such that $\tau = t = 0$. How can we express an external event E in terms of R and G's position?

Traditional textbooks would offer the answer in terms of the Spacetime coordinates of R and G. We an alternative, more intuitive approach for the reader – one which utilizes Python to visualize the behavior of both observers relative to each other. In doing so, not only will the student understand the mechanics of the Galilean Transformations, but also the intuition behind it [15] [14] [13].

To do so, we create the spacetime diagrams of both observers via the Plot.ly library of Python. The prerequisites include creating a scatter trace of text, asking the user for the coordinates of the Event E , velocities of both observers, and setting the range of the figure. The code has been shortened for brevity, but a minimal working example may consist of the following:

```
E_xcoord = int(input ("X-Coordinate of Event: "))
Rob_Vel = int(input ("Rob's Velocity: "))
fig = go.Figure()

fig.add_trace(go.Scatter(
    x=[-8.5,-8.5,.6,8,8,.3,-.3,E_xcoord+.3],
    y=[8,0,1,.7,-.7,8,8,E_ycoord+.5],
    text=["Rob", "Greg",r"$\uptau = t=0$",r"$\uptau$",r"$t$",
    r"$\zeta$",r"$z$",r"$E$"],
    mode="text",
))
fig.update_xaxes(range=[-10, 10])
```

We then plot the Event E from the perspective of Greg:

```
fig.add_trace(
    go.Scatter(
        mode='markers',
        x=[E_xcoord],
        y=[E_ycoord],
        marker=dict(
            color='LightSkyBlue',
            size=15
        ),
        showlegend=False
    )
)
```

Projecting the space-time coordinates of Event E for both Greg and Rob is trivial; for instance, for the space projection of Event E onto Greg's Greek Coordinates, we simply have

```
fig.add_shape(
    type="line",
    x0=0,
    y0=E_ycoord,
    x1=E_xcoord,
    y1=E_ycoord,
    line=dict(
```

```

        color="MediumPurple",
        width=4,
        dash="dot",
    )
)

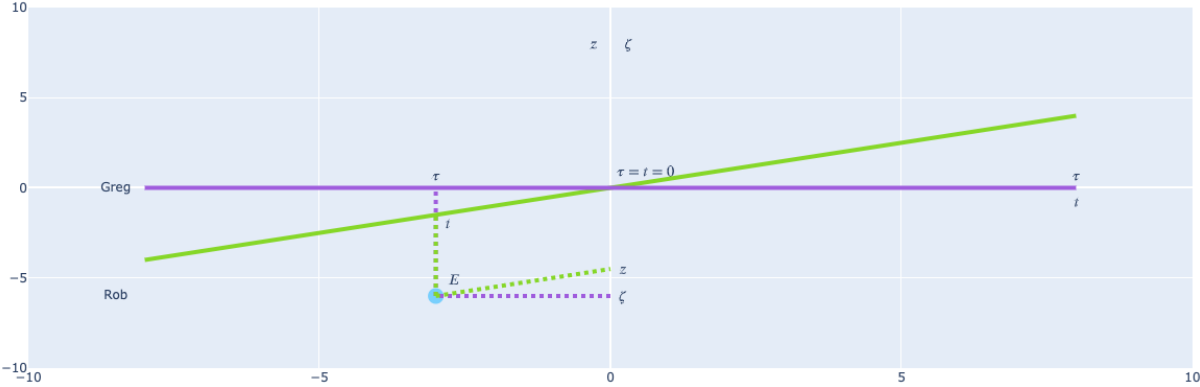
```

What's more interesting are the space-time projections for Rob, which encompass within them the exact Galilean Transformations!

```

fig.add_shape(
    type="line",
    x0=E_xcoord,
    y0=E_ycoord,
    x1=0,
    y1=(E_ycoord-(Rob_Vel*E_xcoord)),
    line=dict(
        color="yellowgreen",
        width=4,
        dash="dot",
    )
)

```



The result is a spacetime diagram of both observers from the reference frame of Greg. An equivalent diagram may be drawn for Rob.

The actual code of the program contains the Galilean Transformations, which may be summarized by the Linear Transformation T as

$$T : \begin{pmatrix} \tau \\ \zeta \end{pmatrix} \rightarrow \begin{pmatrix} t \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix} \begin{pmatrix} \tau \\ \zeta \end{pmatrix} = \begin{pmatrix} \tau \\ v\tau + \zeta \end{pmatrix} = \begin{pmatrix} t \\ z \end{pmatrix} \tag{6}$$

We thus find that both observers share the same time, but witness E at different locations.

$$\tau = t \tag{7}$$

$$v\tau + \zeta = z \tag{8}$$

3.3 Application of GT to Wave Equation

The Wave Equation suffers under the Galilean Transformations, failing to maintain its invariance. To understand why, we must perform the Galilean Transformations on the Wave Equation to verify its invariance. We simplify our analysis by considering a wave dependent only on the z -axis and time.

$$\frac{\partial^2 E}{\partial t^2} = \nabla^2 E \quad (9)$$

$$E_{tt} = E_{xx} + E_{yy} + E_{zz} \quad (10)$$

$$E_{tt} = E_{zz} \quad (11)$$

$$E_{tt} - E_{zz} = 0 \quad (12)$$

We reframe this physical law in the reference frame of a moving observer. The Galilean Transformation has

$$\zeta = z + vt, \tau = t \quad (13)$$

$$E(t, z) = \varepsilon(t, z + vt) = \varepsilon(\tau, \zeta) \quad (14)$$

We use the multivariable chain rule to verify whether $E_{tt} - E_{zz} = 0$. We begin by considering E_t as

$$E(t, z) = \varepsilon(t, z + vt) = \varepsilon(\tau, \zeta) \quad (15)$$

$$E_t = \frac{\partial \varepsilon}{\partial \tau} \cdot \frac{\partial \tau}{\partial t} + \frac{\partial \varepsilon}{\partial \zeta} \cdot \frac{\partial \zeta}{\partial t} \quad (16)$$

$$E_t = \varepsilon_\tau + \varepsilon_\zeta \cdot \frac{\partial(z - vt)}{\partial t} = \varepsilon_\tau - v\varepsilon_\zeta \quad (17)$$

We proceed to finding E_{tt} :

$$E_{tt} = \frac{\partial^2 E}{\partial t^2} = \frac{\partial}{\partial t}(\varepsilon_\tau - v\varepsilon_\zeta) \quad (18)$$

Finding E_{tt} requires evaluating $\frac{\partial}{\partial t}(\varepsilon_\tau)$ and $\frac{\partial}{\partial t}(-v\varepsilon_\zeta)$ as follows

$$\frac{\partial}{\partial t}(\varepsilon_\tau) = \frac{\partial(\varepsilon_\tau)}{\partial \tau} \cdot \frac{\partial \tau}{\partial t} + \frac{\partial(\varepsilon_\tau)}{\partial \zeta} \cdot \frac{\partial \zeta}{\partial t} \quad (19)$$

$$\frac{\partial}{\partial t}(\varepsilon_\tau) = \varepsilon_{\tau\tau} + \varepsilon_{\tau\zeta} \cdot \frac{\partial(z - vt)}{\partial t} = \varepsilon_{\tau\tau} - v\varepsilon_{\tau\zeta} \quad (20)$$

$$\frac{\partial}{\partial t}(-v\varepsilon_\zeta) = \frac{\partial(-v\varepsilon_\zeta)}{\partial \tau} \cdot \frac{\partial \tau}{\partial t} + \frac{\partial(-v\varepsilon_\zeta)}{\partial \zeta} \cdot \frac{\partial \zeta}{\partial t} \quad (21)$$

$$\frac{\partial}{\partial t}(-v\varepsilon_\zeta) = -v\varepsilon_{\zeta\tau} + -v\varepsilon_{\zeta\zeta} \cdot \frac{\partial(z-vt)}{\partial t} = -v\varepsilon_{\zeta\tau} + -v\varepsilon_{\zeta\zeta} \cdot (-v) \quad (22)$$

$$\frac{\partial}{\partial t}(-v\varepsilon_\zeta) = -v\varepsilon_{\zeta\tau} + v^2\varepsilon_{\zeta\zeta} \quad (23)$$

We find that E_{tt} becomes

$$E_{tt} = \frac{\partial}{\partial t}(\varepsilon_\tau) + \frac{\partial}{\partial t}(-v\varepsilon_\zeta) = (\varepsilon_{\tau\tau} - v\varepsilon_{\tau\zeta}) - (-v\varepsilon_{\zeta\tau} + v^2\varepsilon_{\zeta\zeta}) \quad (24)$$

$$E_{tt} = \varepsilon_{\tau\tau} - v\varepsilon_{\tau\zeta} + v\varepsilon_{\zeta\tau} - v^2\varepsilon_{\zeta\zeta} \quad (25)$$

Likewise, we must now find E_{zz} by first considering E_z . Note that $\tau = t = 0$ and that τ has no dependence on z , thus making $\tau_z = 0$ and by extension, $\mathcal{E}_\tau\tau_z = 0$. By the Galilean Transformation, $\zeta = z - vt$, and thus we have

$$E_z = \frac{\partial\varepsilon}{\partial\tau} \cdot \frac{\partial\tau}{\partial z} + \frac{\partial\varepsilon}{\partial\zeta} \cdot \frac{\partial\zeta}{\partial z} \quad (26)$$

$$E_z = \varepsilon_\tau \cdot 0 + \varepsilon_\zeta \cdot \frac{\partial(z-vt)}{\partial z} = \varepsilon_\zeta \quad (27)$$

We take the partial derivative of E_z in respect to z to find E_{zz}

$$E_{zz} = \frac{\partial^2 E}{\partial z^2} = \frac{\partial}{\partial t}(\varepsilon_\zeta) \quad (28)$$

$$\frac{\partial}{\partial t}(\varepsilon_\zeta) = \frac{\partial(\varepsilon_\zeta)}{\partial\tau} \cdot \frac{\partial\tau}{\partial z} + \frac{\partial(\varepsilon_\zeta)}{\partial\zeta} \cdot \frac{\partial\zeta}{\partial z} \quad (29)$$

$$\frac{\partial}{\partial t}(\varepsilon_\zeta) = \varepsilon_{\zeta\tau} \cdot 0 + \varepsilon_{\zeta\zeta} \cdot \frac{\partial(z-vt)}{\partial z} = \varepsilon_{\zeta\zeta} \quad (30)$$

We thus conclude that the Galilean Transformations fail to preserve the Wave Equation as

$$E_{tt} - E_{zz} = \varepsilon_{\tau\tau} - v\varepsilon_{\tau\zeta} + v\varepsilon_{\zeta\tau} - v^2\varepsilon_{\zeta\zeta} - \varepsilon_{\zeta\zeta} \neq 0 \quad (31)$$

3.4 Application of FT to Wave Equation

We thus resort to the Fitzgerald Transformation (FT), coined as an ad-hoc correction [9] to the Galilean Transformation by George Francis Fitzgerald in his 1889 paper on the Ether and the Earth's Atmosphere [8, 30]. Fitzgerald published a colloquial paper, stating that length contracted "by an amount depending on the square of the ratio of their velocities to that of light", which we may translate as

$$E(t, z) = \varepsilon\left(t, \frac{z}{\sqrt{1-v^2}} - \frac{vt}{\sqrt{1-v^2}}\right) = \varepsilon(\tau, \zeta) \quad (32)$$

To verify whether the Fitzgerald Transformations holds the Wave Equation invariant, we must verify whether $E_{tt} - E_{zz} = 0$ by considering E_t as

$$E_t = \frac{\partial \varepsilon}{\partial \tau} \cdot \frac{\partial \tau}{\partial t} + \frac{\partial \varepsilon}{\partial \zeta} \cdot \frac{\partial \zeta}{\partial t} \quad (33)$$

$$E_t = \varepsilon_\tau + \varepsilon_\zeta \cdot \frac{\partial(\frac{z}{\sqrt{1-v^2}} - \frac{vt}{\sqrt{1-v^2}})}{\partial t} = \varepsilon_\tau + \varepsilon_\zeta \cdot \left(-\frac{v}{\sqrt{1-v^2}}\right) \quad (34)$$

$$E_t = \varepsilon_\tau - \frac{v\varepsilon_\zeta}{\sqrt{1-v^2}} \quad (35)$$

Finding E_{tt} will require an extensive breakdown of each of E_t 's terms by taking the partial derivative of each term in respect to time, using the multivariable chain rule.

$$E_{tt} = \frac{\partial^2 E}{\partial t^2} = \frac{\partial}{\partial t} \left(\varepsilon_\tau - \frac{v\varepsilon_\zeta}{\sqrt{1-v^2}} \right) \quad (36)$$

We begin by finding the partial derivative of the ε_τ term in respect to time:

$$\frac{\partial}{\partial t}(\varepsilon_\tau) = \frac{\partial(\varepsilon_\tau)}{\partial \tau} \cdot \frac{\partial \tau}{\partial t} + \frac{\partial(\varepsilon_\tau)}{\partial \zeta} \cdot \frac{\partial \zeta}{\partial t} \quad (37)$$

$$\frac{\partial}{\partial t}(\varepsilon_\tau) = \varepsilon_{\tau\tau} + \varepsilon_{\tau\zeta} \cdot \frac{\partial(\frac{z}{\sqrt{1-v^2}} - \frac{vt}{\sqrt{1-v^2}})}{\partial t} = \varepsilon_{\tau\tau} + \varepsilon_{\tau\zeta} \cdot \left(-\frac{v}{\sqrt{1-v^2}}\right) \quad (38)$$

$$\frac{\partial}{\partial t}(\varepsilon_\tau) = \varepsilon_{\tau\tau} - \frac{v\varepsilon_{\tau\zeta}}{\sqrt{1-v^2}} \quad (39)$$

We proceed to evaluating the partial derivative of $-\frac{v\varepsilon_\zeta}{\sqrt{1-v^2}}$ in respect to time

$$\frac{\partial}{\partial t} \left(-\frac{v\varepsilon_\zeta}{\sqrt{1-v^2}} \right) = \frac{\partial \left(-\frac{v\varepsilon_\zeta}{\sqrt{1-v^2}} \right)}{\partial \tau} \cdot \frac{\partial \tau}{\partial t} + \frac{\partial \left(-\frac{v\varepsilon_\zeta}{\sqrt{1-v^2}} \right)}{\partial \zeta} \cdot \frac{\partial \zeta}{\partial t} \quad (40)$$

$$\frac{\partial}{\partial t} \left(-\frac{v\varepsilon_\zeta}{\sqrt{1-v^2}} \right) = -\frac{v\varepsilon_{\zeta\tau}}{\sqrt{1-v^2}} + \left(-\frac{v\varepsilon_{\zeta\zeta}}{\sqrt{1-v^2}} \cdot \frac{\partial(\frac{z}{\sqrt{1-v^2}} - \frac{vt}{\sqrt{1-v^2}})}{\partial t} \right) \quad (41)$$

$$\frac{\partial}{\partial t} \left(-\frac{v\varepsilon_\zeta}{\sqrt{1-v^2}} \right) = -\frac{v\varepsilon_{\zeta\tau}}{\sqrt{1-v^2}} + \left(-\frac{v\varepsilon_{\zeta\zeta}}{\sqrt{1-v^2}} \cdot \left(-\frac{v}{\sqrt{1-v^2}}\right) \right) = -\frac{v\varepsilon_{\zeta\tau}}{\sqrt{1-v^2}} + \frac{v^2\varepsilon_{\zeta\zeta}}{1-v^2} \quad (42)$$

We can thus conclude that the second degree partial derivative of the Electric Field in respect to time is

$$E_{tt} = \frac{\partial}{\partial t}(\varepsilon_\tau) + \frac{\partial}{\partial t} \left(-\frac{v\varepsilon_\zeta}{\sqrt{1-v^2}} \right) = \varepsilon_{\tau\tau} - \frac{v\varepsilon_{\tau\zeta}}{\sqrt{1-v^2}} - \frac{v\varepsilon_{\zeta\tau}}{\sqrt{1-v^2}} + \frac{v^2\varepsilon_{\zeta\zeta}}{1-v^2} \quad (43)$$

$$= \varepsilon_{\tau\tau} - \frac{2v\varepsilon_{\tau\zeta}}{\sqrt{1-v^2}} + \frac{v^2\varepsilon_{\zeta\zeta}}{1-v^2} \quad (44)$$

We repeat the process for finding E_z , which we can expand via the multivariable chain rule as

$$E_z = \frac{\partial \varepsilon}{\partial \tau} \cdot \frac{\partial \tau}{\partial z} + \frac{\partial \varepsilon}{\partial \zeta} \cdot \frac{\partial \zeta}{\partial z} \quad (45)$$

In doing so, we easily compute E_z to be

$$E_z = \varepsilon_\tau \cdot 0 + \varepsilon_\zeta \cdot \frac{\partial(\frac{z}{\sqrt{1-v^2}} - \frac{vt}{\sqrt{1-v^2}})}{\partial z} = \frac{\varepsilon_\zeta}{\sqrt{1-v^2}} \quad (46)$$

All that remains is to find E_{zz} by computing a few more partial derivatives.

$$\frac{\partial}{\partial z}(E_z) = \frac{\partial^2 E}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\varepsilon_\zeta}{\sqrt{1-v^2}}\right) \quad (47)$$

$$\frac{\partial}{\partial z}\left(\frac{\varepsilon_\zeta}{\sqrt{1-v^2}}\right) = \frac{\partial(\frac{\varepsilon_\zeta}{\sqrt{1-v^2}})}{\partial \tau} \cdot \frac{\partial \tau}{\partial z} + \frac{\partial(\frac{\varepsilon_\zeta}{\sqrt{1-v^2}})}{\partial \zeta} \cdot \frac{\partial \zeta}{\partial z} \quad (48)$$

$$\frac{\partial}{\partial z}\left(\frac{\varepsilon_\zeta}{\sqrt{1-v^2}}\right) = \frac{\varepsilon_{\zeta\tau}}{\sqrt{1-v^2}} \cdot 0 + \frac{\varepsilon_{\zeta\zeta}}{\sqrt{1-v^2}} \cdot \frac{\partial(\frac{z}{\sqrt{1-v^2}} - \frac{vt}{\sqrt{1-v^2}})}{\partial z} \quad (49)$$

$$\frac{\partial}{\partial z}\left(\frac{\varepsilon_\zeta}{\sqrt{1-v^2}}\right) = \frac{\varepsilon_{\zeta\zeta}}{\sqrt{1-v^2}} \cdot \frac{1}{\sqrt{1-v^2}} = \frac{\varepsilon_{\zeta\zeta}}{1-v^2} \quad (50)$$

$$E_{zz} = \frac{\varepsilon_{\zeta\zeta}}{1-v^2} \quad (51)$$

Finally, we may verify whether the Fitzgerald Transformation does indeed preserve the invariance of the Wave Equation by checking whether $E_{tt} - E_{zz} = 0$ via simple substitution:

$$E_{tt} - E_{zz} = \left(\varepsilon_{\tau\tau} - \frac{2v\varepsilon_{\tau\zeta}}{\sqrt{1-v^2}} + \frac{v^2\varepsilon_{\zeta\zeta}}{1-v^2}\right) - \left(\frac{\varepsilon_{\zeta\zeta}}{1-v^2}\right) = \varepsilon_{\tau\tau} - \frac{2v\varepsilon_{\tau\zeta}}{\sqrt{1-v^2}} + \frac{\varepsilon_{\zeta\zeta}(v^2-1)}{1-v^2} \quad (52)$$

$$E_{tt} - E_{zz} = \varepsilon_{\tau\tau} - \frac{2v\varepsilon_{\tau\zeta}}{\sqrt{1-v^2}} + \frac{\varepsilon_{\zeta\zeta}(v-1)(v+1)}{(1-v)(1+v)} = \varepsilon_{\tau\tau} - \varepsilon_{\zeta\zeta} - \frac{2v\varepsilon_{\tau\zeta}}{\sqrt{1-v^2}} \neq 0 \quad (53)$$

Alas, even the Fitzgerald Transformation has failed to correct the Galilean Transformation. And yet we may retain some semblance of hope upon examining the matrix analogue of the Fitzgerald Transformation

$$F : R^2 \rightarrow R^2 = \begin{pmatrix} 1 & 0 \\ -v & 1 \\ \sqrt{1-v^2} & \sqrt{1-v^2} \end{pmatrix} \begin{pmatrix} t \\ z \end{pmatrix} = \begin{pmatrix} t \\ \frac{z-vt}{\sqrt{1-v^2}} \end{pmatrix} = \begin{pmatrix} \tau \\ \zeta \end{pmatrix} \quad (54)$$

It's worth examining the structure of F . The Galilean Transformations held everything same for all observers – time, length, velocity. George Francis Fitzgerald took a leap and made the unconventional wager that not all observers measure lengths the same, likely inspired by his Electrical Engineer friend Oliver Heaviside [7] – who also condensed Maxwell's 16 Equations into the 4 Macroscopic ones we know today. Hendrik Lorentz independently made the next leap – an even bolder claim that time itself was subject to relativistic changes, via the Lorentz Transformation

$$L : R^2 \rightarrow R^2 : \begin{pmatrix} \frac{1}{\sqrt{1-v^2}} & \frac{-v}{\sqrt{1-v^2}} \\ \frac{-v}{\sqrt{1-v^2}} & \frac{1}{\sqrt{1-v^2}} \end{pmatrix} \quad (55)$$

Immediately, we find the transformation to be symmetric such that $L = L^T$ and $L = L^{-1}$. But does it maintain the invariance of the wave equation? To verify, we must execute the transformation on the (t, z) coordinates of one observer's reference frame.

3.5 Application of LT to Wave Equation

We now consider the Lorentz Transformations (LT) and test whether the Wave Equation holds invariant under L .

$$L : R^2 \rightarrow R^2 : \begin{pmatrix} \frac{1}{\sqrt{1-v^2}} & \frac{-v}{\sqrt{1-v^2}} \\ \frac{-v}{\sqrt{1-v^2}} & \frac{1}{\sqrt{1-v^2}} \end{pmatrix} \begin{pmatrix} t \\ z \end{pmatrix} = \begin{pmatrix} \frac{t-vz}{\sqrt{1-v^2}} \\ \frac{z-vt}{\sqrt{1-v^2}} \end{pmatrix} \quad (56)$$

$$E(t, z) = \varepsilon(\tau, \zeta) = \varepsilon\left(\frac{t-vz}{\sqrt{1-v^2}}, \frac{z-vt}{\sqrt{1-v^2}}\right) \quad (57)$$

We consider once again the axiom $E_{tt} - E_{zz} = 0$ and validate the Lorentz Transformations in light of the wave equation by first considering E_t as

$$E_t = \frac{\partial \varepsilon}{\partial \tau} \cdot \frac{\partial \tau}{\partial t} + \frac{\partial \varepsilon}{\partial \zeta} \cdot \frac{\partial \zeta}{\partial t} \quad (58)$$

The two observers' times are no longer equivalent. Indeed, by the Lorentz Transformation, we now have time dilated as one's velocity v approaches the speed of light c .

$$E_t = \varepsilon_\tau \cdot \frac{\partial\left(\frac{t}{\sqrt{1-v^2}} - \frac{vz}{\sqrt{1-v^2}}\right)}{\partial t} + \varepsilon_\zeta \cdot \frac{\partial\left(\frac{z}{\sqrt{1-v^2}} - \frac{vt}{\sqrt{1-v^2}}\right)}{\partial t} \quad (59)$$

$$E_t = \varepsilon_\tau \cdot \frac{1}{\sqrt{1-v^2}} + \varepsilon_\zeta \cdot \left(-\frac{v}{\sqrt{1-v^2}}\right) = \frac{\varepsilon_\tau - \varepsilon_\zeta v}{\sqrt{1-v^2}} \quad (60)$$

Taking the second partial derivative of the Electric Field in respect to time, we have

$$E_{tt} = \frac{\partial}{\partial \tau}(E_t) = \frac{\partial}{\partial \tau}\left(\frac{\varepsilon_\tau - \varepsilon_\zeta v}{\sqrt{1-v^2}}\right) \quad (61)$$

$$\frac{\partial}{\partial \tau}\left(\frac{\varepsilon_\tau - \varepsilon_\zeta v}{\sqrt{1-v^2}}\right) = \frac{\partial\left(\frac{\varepsilon_\tau - \varepsilon_\zeta v}{\sqrt{1-v^2}}\right)}{\partial \tau} \cdot \frac{\partial \tau}{\partial t} + \frac{\partial\left(\frac{\varepsilon_\tau - \varepsilon_\zeta v}{\sqrt{1-v^2}}\right)}{\partial \zeta} \cdot \frac{\partial \zeta}{\partial t} \quad (62)$$

Recall again that time and length are now relativistic, and must be duly replaced with their respective expressions

$$\frac{\partial}{\partial \tau}\left(\frac{\varepsilon_\tau - \varepsilon_\zeta v}{\sqrt{1-v^2}}\right) = \frac{\varepsilon_{\tau\tau} - \varepsilon_{\zeta\tau}v}{\sqrt{1-v^2}} \cdot \frac{\partial\left(\frac{t}{\sqrt{1-v^2}} - \frac{vz}{\sqrt{1-v^2}}\right)}{\partial t} + \frac{\varepsilon_{\tau\zeta} - \varepsilon_{\zeta\zeta}v}{\sqrt{1-v^2}} \cdot \frac{\partial\left(\frac{z}{\sqrt{1-v^2}} - \frac{vt}{\sqrt{1-v^2}}\right)}{\partial t} \quad (63)$$

$$\frac{\partial}{\partial \tau}\left(\frac{\varepsilon_\tau - \varepsilon_\zeta v}{\sqrt{1-v^2}}\right) = \frac{\varepsilon_{\tau\tau} - \varepsilon_{\zeta\tau}v}{\sqrt{1-v^2}} \cdot \frac{1}{\sqrt{1-v^2}} + \frac{\varepsilon_{\tau\zeta} - \varepsilon_{\zeta\zeta}v}{\sqrt{1-v^2}} \cdot \frac{-v}{\sqrt{1-v^2}} \quad (64)$$

$$= \frac{\varepsilon_{\tau\tau} - \varepsilon_{\zeta\tau}v}{1-v^2} + \frac{-v(\varepsilon_{\tau\zeta} - \varepsilon_{\zeta\zeta}v)}{1-v^2} \quad (65)$$

We conclude that E_{tt} is

$$E_{tt} = \frac{\partial}{\partial \tau}\left(\frac{\varepsilon_\tau - \varepsilon_\zeta v}{\sqrt{1-v^2}}\right) = \frac{\varepsilon_{\tau\tau} - \varepsilon_{\zeta\tau}v}{1-v^2} + \frac{-v\varepsilon_{\tau\zeta} + \varepsilon_{\zeta\zeta}v^2}{1-v^2} = \frac{\varepsilon_{\tau\tau} - 2\varepsilon_{\zeta\tau}v + \varepsilon_{\zeta\zeta}v^2}{1-v^2} \quad (66)$$

We now have only E_{zz} to derive, which we must begin by finding E_z . We begin by expanding via the multivariable chain rule as follows:

$$E_z = \frac{\partial \varepsilon}{\partial \tau} \cdot \frac{\partial \tau}{\partial z} + \frac{\partial \varepsilon}{\partial \zeta} \cdot \frac{\partial \zeta}{\partial z} \quad (67)$$

We expand the expression to find E_z

$$E_z = \varepsilon_\tau \cdot \frac{\partial(\frac{t}{\sqrt{1-v^2}} - \frac{vz}{\sqrt{1-v^2}})}{\partial z} + \varepsilon_\zeta \cdot \frac{\partial(\frac{z}{\sqrt{1-v^2}} - \frac{vt}{\sqrt{1-v^2}})}{\partial z} \quad (68)$$

$$E_z = \varepsilon_\tau \cdot -\frac{v}{\sqrt{1-v^2}} + \varepsilon_\zeta \cdot \frac{1}{\sqrt{1-v^2}} = \frac{\varepsilon_\zeta - v\varepsilon_\tau}{\sqrt{1-v^2}} \quad (69)$$

Now we simply find the second partial derivative of the Electric Field in respect to z by expanding with the chain rule

$$E_{zz} = \frac{\partial}{\partial z} \left(\frac{\varepsilon_\zeta - v\varepsilon_\tau}{\sqrt{1-v^2}} \right) = \frac{\partial(\frac{\varepsilon_\zeta - v\varepsilon_\tau}{\sqrt{1-v^2}})}{\partial \tau} \cdot \frac{\partial \tau}{\partial z} + \frac{\partial(\frac{\varepsilon_\zeta - v\varepsilon_\tau}{\sqrt{1-v^2}})}{\partial \zeta} \cdot \frac{\partial \zeta}{\partial z} \quad (70)$$

We continue simplifying to find E_{zz} to be

$$E_{zz} = \frac{\partial}{\partial z} \left(\frac{\varepsilon_\zeta - v\varepsilon_\tau}{\sqrt{1-v^2}} \right) = \frac{\partial(\frac{\varepsilon_\zeta - v\varepsilon_\tau}{\sqrt{1-v^2}})}{\partial \tau} \cdot \frac{\partial \tau}{\partial z} + \frac{\partial(\frac{\varepsilon_\zeta - v\varepsilon_\tau}{\sqrt{1-v^2}})}{\partial \zeta} \cdot \frac{\partial \zeta}{\partial z} \quad (71)$$

We expand the time τ and z-axis ζ of the second observer in motion to have

$$E_{zz} = \frac{\varepsilon_{\zeta\tau} - v\varepsilon_{\tau\tau}}{\sqrt{1-v^2}} \cdot \frac{\partial(\frac{t}{\sqrt{1-v^2}} - \frac{vz}{\sqrt{1-v^2}})}{\partial z} + \frac{\varepsilon_{\zeta\zeta} - v\varepsilon_{\tau\zeta}}{\sqrt{1-v^2}} \cdot \frac{\partial(\frac{z}{\sqrt{1-v^2}} - \frac{vt}{\sqrt{1-v^2}})}{\partial z} \quad (72)$$

Simplifying our partial derivatives, we have

$$E_{zz} = \frac{\varepsilon_{\zeta\tau} - v\varepsilon_{\tau\tau}}{\sqrt{1-v^2}} \cdot -\frac{v}{\sqrt{1-v^2}} + \frac{\varepsilon_{\zeta\zeta} - v\varepsilon_{\tau\zeta}}{\sqrt{1-v^2}} \cdot \frac{1}{\sqrt{1-v^2}} = \frac{-v(\varepsilon_{\zeta\tau} - v\varepsilon_{\tau\tau})}{1-v^2} + \frac{\varepsilon_{\zeta\zeta} - v\varepsilon_{\tau\zeta}}{1-v^2} \quad (73)$$

Alas, we find E_{zz} to be

$$E_{zz} = \frac{-v\varepsilon_{\zeta\tau} + v^2\varepsilon_{\tau\tau}}{1-v^2} + \frac{\varepsilon_{\zeta\zeta} - v\varepsilon_{\tau\zeta}}{1-v^2} = \frac{-2v\varepsilon_{\zeta\tau} + v^2\varepsilon_{\tau\tau} + \varepsilon_{\zeta\zeta}}{1-v^2} \quad (74)$$

Wrapping up the proof, we find that

$$E_{tt} - E_{zz} = \frac{\varepsilon_{\tau\tau} - 2\varepsilon_{\zeta\tau}v + \varepsilon_{\zeta\zeta}v^2}{1-v^2} - \frac{-2v\varepsilon_{\zeta\tau} + v^2\varepsilon_{\tau\tau} + \varepsilon_{\zeta\zeta}}{1-v^2} = 0 \quad (75)$$

We have just proved the Wave Equation does indeed hold invariant under the Lorentz Transformation L . From this transformation alone, rises all major tenets of Special Relativity, including Time Dilation and Length Contraction.

3.6 Applications of LT

To observe the utility of the Lorentz Transformations, we need only revert back from light units ($c = 1$) such that $v = \frac{v}{c}$. By the Lorentz Transformation we just derived, we have the Length Contraction as

$$\Delta x' = x'_2 - x'_1 = \frac{z - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (76)$$

Traditionally, we reduce the numerator to some x in the stationary observer's reference frame such that the contracted length may be expressed as

$$\Delta x' = \frac{x}{\gamma} \quad (77)$$

where $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$. Likewise, we may condense the time dilation formula as

$$\Delta t' = t'_2 - t'_1 = \frac{t - \frac{vz}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (78)$$

$$\Delta t' = \gamma \Delta t \quad (79)$$

Note the nature of the Time Dilation and Length Contraction formulas. We find the γ term in the numerator for the former and in the denominator in the latter, since one quantity dilates and the other contracts. Indeed, this makes it quite easy to see that when $v \ll c$, γ becomes 1 and the expressions depress to their simpler, Galilean counterparts.

By simply dividing them, we can obtain the velocity transformation equation under the Lorentz Transformation. In the instance we seek to describe the velocity of an event in the reference frame of another observer, we use this new, refined velocity addition formula – as opposed to the traditional, Newtonian sum of velocities

$$\frac{\Delta x'}{\Delta t'} = \frac{z - vt}{t - \frac{vz}{c^2}} \cdot \frac{\frac{1}{t}}{\frac{1}{t}} = \frac{\frac{z}{t} - \frac{vt}{t}}{1 - \frac{vz}{c^2 t}} = \frac{u - v}{1 - \frac{uv}{c^2}} \quad (80)$$

Indeed, we can verify the constancy of the speed of light c by simply letting $u = c$ as

$$j = \frac{u - v}{1 - \frac{uv}{c^2}} = \frac{c - v}{1 - \frac{cv}{c^2}} = \frac{c - v}{1 - \frac{v}{c}} = \frac{c - v}{\frac{c - v}{c}} = c \quad (81)$$

And indeed, we find that the speed of light remains constant, even for the stationary observer.

In this section, we have provided a visual, interactive computational program for demonstrating the Galilean Transformations, which takes the user's input for the velocities of both observers and the coordinates of the event E and constructs a visualization of the space-time diagram for both observers in respect to the GT. Thereafter, the Galilean Transformations were tested on the Wave Equation, but failed to maintain its invariance, after which the Fitzgerald and Lorentz Transformations were tested. Finally, a few brief applications of the LT were provided, including the equations for Length Contraction, Time Dilation, and proof of the Constancy of the Speed of Light. We now consider the second major principle of Relativity that can be derived from Maxwell's Equations – namely, the effect of Time Dilation. We make this derivation explicit using a computational program that demonstrates the Electromagnetic situation at hand.

4 Time Dilation from EM

Special Relativity has a whole host of effects, from Relativistic Mass and Length Contraction to Time Dilation. Many of these effects seem out of reach for an undergraduate learning Electrodynamics. And yet, the very trails of Special Relativity arises from a classic Electromagnetic Scenarios – a point charge and flat sheet in motion.

After understanding the aforementioned EM Scenario with Gauss' Law and Ampere's Law, we seek to construct a computational program that demonstrates the velocity transformation and time dilation that occurs within this problem. We then conclude by considering further consequences of this simple problem.

4.1 Charge and Sheet in Motion

We begin by considering the following scenario:

A Charge Q moves in the $+\hat{i}$ at a constant velocity v alongside a negatively charged sheet moving at the same velocity in the same direction.

The question seems trivial: how will the Charge Q move? Will it fall straight into the charged sheet, due to the positive electrostatic attraction? The answer – counter-intuitively – is far from a predictable linear path. In fact, the journey to the answer will uncover one of the effects of Special Relativity itself – Time Dilation. This is no accident – indeed, situations involving relativity can be expected when encountering even the simplest of Electromagnetic scenarios.

Understanding the behavior of Q begins with a Free Body Diagram involving all forces involved. We first consider the magnitude of the Electric Field of an infinite sheet of charge. To do so, we utilize Gauss' Law as

$$\int E \cdot dA = \frac{q_{enc}}{\epsilon_0} \quad (82)$$

We construct a Gaussian Surface which respects the rotational symmetry of the Electric Field of the Infinite Sheet of Charge. This would be a cylinder, which gives

$$E \cdot (2A) = \frac{\sigma A}{\epsilon_0} \quad (83)$$

$$E = \frac{\sigma}{2\epsilon_0} \quad (84)$$

Being oppositely charged, the direction of the Electric Field would of course be $+\hat{j}$. We consider now the Electric Force, which is simply

$$F_E = EQ = \frac{Q\sigma}{2\epsilon_0} \quad (85)$$

As $F_E \sim E$, we have $\text{dir}(F_E) = \text{dir}(E)$. We now consider the Magnetic Field with Ampere's Law as

$$\int B \cdot dS = J \cdot I_{enc} \quad (86)$$

Once again, we do so in respect to the symmetry of the field, which informs our decision to choose a rectangle as our Amperian Loop. Two lengths of the rectangle will be perpendicular to the magnetic field, and thus make no contribution to it, and two lengths will be parallel, which gives

$$B \cdot (2h) = J(\mu_0 h) \quad (87)$$

$$B = \frac{J\mu_0}{2} \quad (88)$$

We have $\text{dir}(B) = \hat{k}$ due to the Right Hand Rule (RHR). We may dissect the infinite sheet of charges as a set of infinite wires, each with a significant magnetic field contribution. Applying the RHR, we find the Magnetic Field at Q to be counterclockwise. Recall, however, we have a negative sheet and the Right Hand Rule applies for positive charges or conventional current. We thus reverse our direction and conclude $\text{dir}(B) = \hat{k}$.

We consider the magnetic force as $F_B = Q(v \times B)$, and have $\text{dir}(F_B) = -\hat{j}$ by the well-defined direction of the Cross Product. We further have an initial velocity with $\text{dir}(v) = \hat{i}$.

Considering the Net Force on the Test Charge Q in conjunction with Newton's Second Law provides us the equation for the charge's path as

$$\begin{aligned} \vec{F} &= Q(\vec{E} + \vec{v} \times \vec{B}) \\ \vec{F} &= Q\left(\frac{\sigma}{2\epsilon_0} - \vec{v} \times \frac{\mu_0\sigma_s}{2}\right) \rightarrow \vec{F} = Q\left(\frac{\sigma}{2\epsilon_0} - \frac{\mu_0 v^2 \sigma}{2}\right). \end{aligned} \quad (89)$$

Which we can simplify further to be

$$F = \frac{Q\sigma}{2\epsilon_0} (1 - \mu_0\epsilon_0 v^2) = \frac{Q\sigma}{2\epsilon_0} \left(1 - \frac{v^2}{c^2}\right) = QE \left(1 - \frac{v^2}{c^2}\right) \quad (90)$$

in which we utilized two key computations to simplify the expression for $\vec{r}(t)$ – simplifying $\frac{1}{(\mu_0\epsilon_0)^2}$ as c^2 . We also used the formula derived for the Constant Electric Field of an Infinite Sheet of Charge as $E = \frac{\sigma}{2\epsilon_0}$ to simplify the expression.

For a Stationary Observer, there exists both an Electric Field produced by the Charge Q and a Magnetic Field due to it's initial movement along the x-axis. For an observer moving at a velocity v with the Charged Sheet, there only exists an Electric Field $F = QE$, produced by the stationary, nonmoving charge. This results in the inner term of the velocity term going to unity, thus producing a smaller time for the moving observer. This becomes clear as

$$t = \sqrt{\frac{2mr(t)}{QE \left(1 - \frac{v^2}{c^2}\right)}} \quad (91)$$

for the Stationary Observer. For the moving observer, however, the second term in the denominator becomes unity, resulting in a smaller perceived time for the charge Q to traverse the same distance due to a larger velocity v

$$t' = \sqrt{\frac{2mr(t)}{QE}} = \sqrt{\frac{2mr(t)}{QE \left(1 - \frac{v^2}{c^2}\right)}} \left(\sqrt{\left(1 - \frac{v^2}{c^2}\right)}\right) = t \sqrt{\left(1 - \frac{v^2}{c^2}\right)} \quad (92)$$

Indeed, in considering the most typical of Electromagnetic scenarios, we have derived the Lorentz Gamma Factor γ . What a result!

4.2 Visualizing Motion of Charge in EM Field

And yet, we can take this idea a degree further by actually visualizing the movement of the Test Charge Q , using Computational Programming. We begin by simply constructing a canvas in which we'll draw the Sheet of Charge and its companion Test Charge, making sure to declare the version of programming language we use and create a canvas suitable for our visualization.

```
GlowScript 3.0 VPython
scene2 = canvas(width=1400, height=500, center=vector(2.5e-8,2.5e-8,0),
↪ background=color.white)
```

Before proceeding, it's worth creating a physical representation of the $\hat{i}, \hat{j}, \hat{k}$ directions, in the form of the x, y, z axes. These will come in play later as we consider the direction of the different forces on the Test Charge Q .

```
for i in range (-5,5):
    rate(4)
    axesX = box(length = .5e-8, width = 0.2*(1e-8), height = .5e-8, pos =
↪ vector(i*(1e-8), 0, 0), color = color.green, opacity = 0.2)
    axesY = box(length = 0.2*(1e-8), width = .5e-8, height = .5e-8, pos =
↪ vector(0, i*(1e-8), 0), color = color.magenta, opacity = 0.2)
    axesZ = box(length = .5e-8, width = .5e-8, height = 0.2*(1e-8), pos =
↪ vector(0, 0, i*(1e-8)), color = color.cyan, opacity = 0.2)
```

Note the atomic scale of both the canvas and the axes – this is simply to make the scale realistic. In the interest of saving time in the future, we declare all useful constants, including a Scale Factor which will be occasionally used. As this is a 3D environment, it's worth setting up effective lighting for the scene, in the form of a 'lamp' and a sphere embodying the lamp's luminous intensity as

```
lamp = local_light(pos=vector(3e-8,3e-8,-2.5e-8), color=color.yellow)
sphere(pos = vector(3e-8,3e-8,-2.5e-8), emissive = True,
radius = 1e-9, color = color.yellow)
```

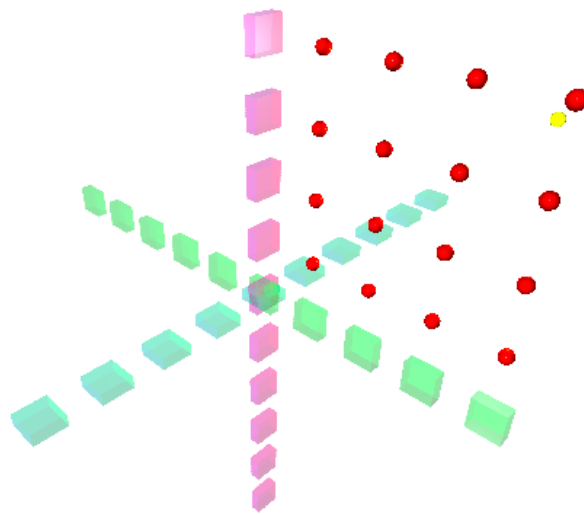
First and foremost, we consider the Infinite Sheet of Negative Charge. For the sake of our purposes and in the interest of saving rendering time, we will utilize a visibly finite sheet for the time being – keeping in mind, however, that the sheet is ideally infinite. We utilize a nested for loop as

```
for i in range (1,5):
    for j in range (1,5):
        for k in range (1,5):
            rate (50)
            sources.append(sphere (pos = vector(i*(1e-8),j*(1e-8),0),
↪ radius = 1e-9, color = color.red, q = -Q))
```



```
r_obs.append(vector(i*(1e-8),j*(1e-8),k*(1e-8)))
r_obs2.append(vector(i*(1e-8),j*(1e-8),-k*(1e-8)))
```

The for loop uses two of its three variables to construct a two-dimensional, flat sheet of infinite charge. To indicate the value of the charge, we've defined an attribute q , such that any charge in `sources[]` has `sources[i].q=-Q`, indicative of a negative charge. The loop creates a charge and appends it to the array `sources[]`. We also have observation points being appended to `r_obs` and `r_obs2`. The loop appends an observation point to either one of these two sets – the two sets represent the front and back, respectively, of the sheet of charge.



We now take every single charge in `sources[]` and consider them as one flat sheet of negative charge by combining them via the `compound` function. To do so, we loop through the elements of `sources[]` and append each one to a new placeholder array, aptly titled by the same name, which will then be compounded into a single sheet of charge.

```
for i in range(0,len(sources)):
    placeholder.append(sources[i])
charges = compound(placeholder)
```

We now consider the test charge, which can be easily created using the `sphere()` function. We give it a radius suitable for a Proton, although in reality the Proton has a radius to the order of 10^{-15} , a number contested due to the proton-size puzzle [31], which is in regards to a minor discrepancy in the size of the Proton. We also assign a trail to the Test Charge in the interest of following its trajectory.

```
testCharge = sphere(pos = vector(2.5e-8,-2.5e-8,-2e-8), radius = 1e-9,
↳ make_trail = True, trail_radius = .3*1e-9, trail_color = color.purple,
↳ emissive = True)
```

We proceed to generating the actual electric field of the Sheet of Charge via Coulomb's Law as

```
for i in range (0, len(sources)):
    rate(20)
    print("sources.pos = " + sources[i].pos)

    E.append(vector(0,0,0))
    E2.append(vector(0,0,0))

    print("r_obs[i] is "+r_obs[i])
    print("sources[i].pos is "+sources[i].pos)
    print("Their difference r_obs[i]-sources[i].pos is "+r_obs[i])

    r.append(r_obs[i]-sources[i].pos)
    r2.append(r_obs2[i]-sources[i].pos)

    E[i] = E[i] + ((k * Q)/(mag(r[i])**2)) * (-norm(r[i]))
    E2[i] = E2[i] + ((k * Q)/(mag(r2[i])**2)) * (-norm(r2[i]))

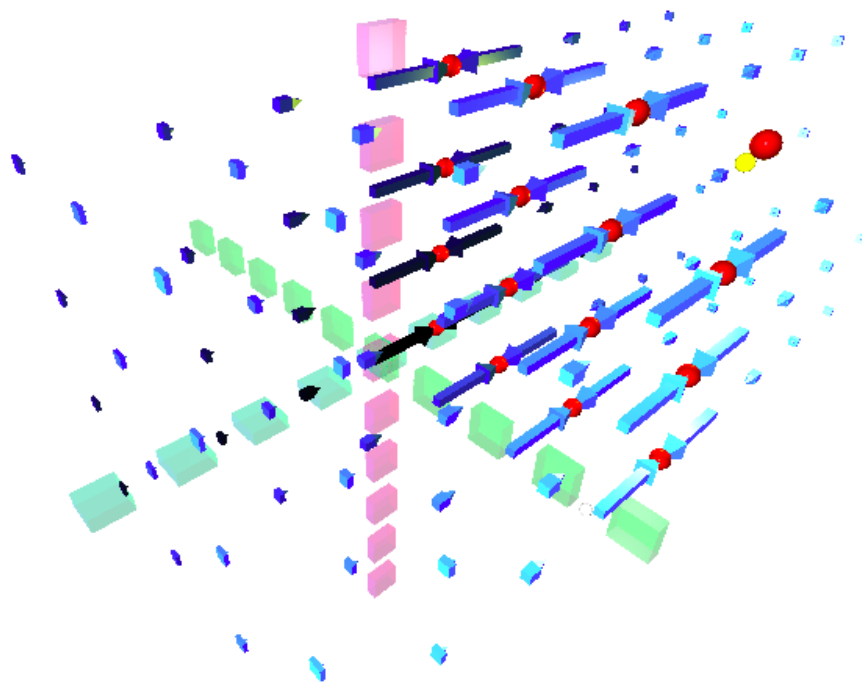
    FieldVectors.append(arrow (pos = r_obs[i], color =
↳ vec(i/100,i/50,i/10), axis = ScaleFactor*E[i], shaftwidth =
↳ sources[i].radius, opacity = mag(r[i])))
    FieldVectors2.append(arrow (pos = r_obs2[i], color =
↳ vec(i/100,i/50,i/10), axis = ScaleFactor*E2[i], shaftwidth =
↳ sources[i].radius, opacity = mag(r[i])))

    print("FieldVectors.pos = " + FieldVectors[i].pos)
```

We thus conclude $\exists E_1, E_2$, one on either side of the flat sheet of charge. We begin by initializing the Electric Field on both sides to be `vector(0,0,0)`. We then consider our radius r to be the difference between our observation point and the actual source charge in the sheet, and append those radii to their respective sets. The magnitude of the electric field vectors are then computed via Coulomb's Law, and their magnitude is taken by performing the `norm()` function on the radii. Occasional breakpoints such as the one below may be useful in diagnosing errors in the magnitude of the Electric Field, should they arise.

```
print("Why is E[i] = NaN?) r[i] = " + r[i])
```

Once again, we seek to combine all the Electric Field vectors on either side of the Sheet of Charge using the `compound()` function as needed



```

for i in range(0,len(FieldVectors)):
    FieldVectorsPlaceholder.append(FieldVectors[i])
    SecondFieldVectorsPlaceholder.append(FieldVectors2[i])

movingVectors = compound(FieldVectorsPlaceholder)
secondMovingVectors = compound(SecondFieldVectorsPlaceholder)

```

The utility in combining these elements will become evident as we seek to increment their positions. For the time being, however, it is far easier to manipulate the elements as a homogeneous group of vectors rather than isolated parts.

We now consider the forces acting on our test charge Q . First and foremost, we consider the electric field at the test charge. To do so, we compute the Electric Field due to every single component of the sheet of charge at Q , and use a loop to append all the vectors to their respective arrays as

```

for i in range (0, len(sources)):
    rate(20)
    E_TestCharge.append(testCharge.pos)
    r_TestCharge.append(testCharge.pos-sources[i].pos)
    E_TestCharge[i] = E_TestCharge[i] + ((k *
    ↪ Q)/(mag(r_TestCharge[i])**2)) * (-norm(r_TestCharge[i]))

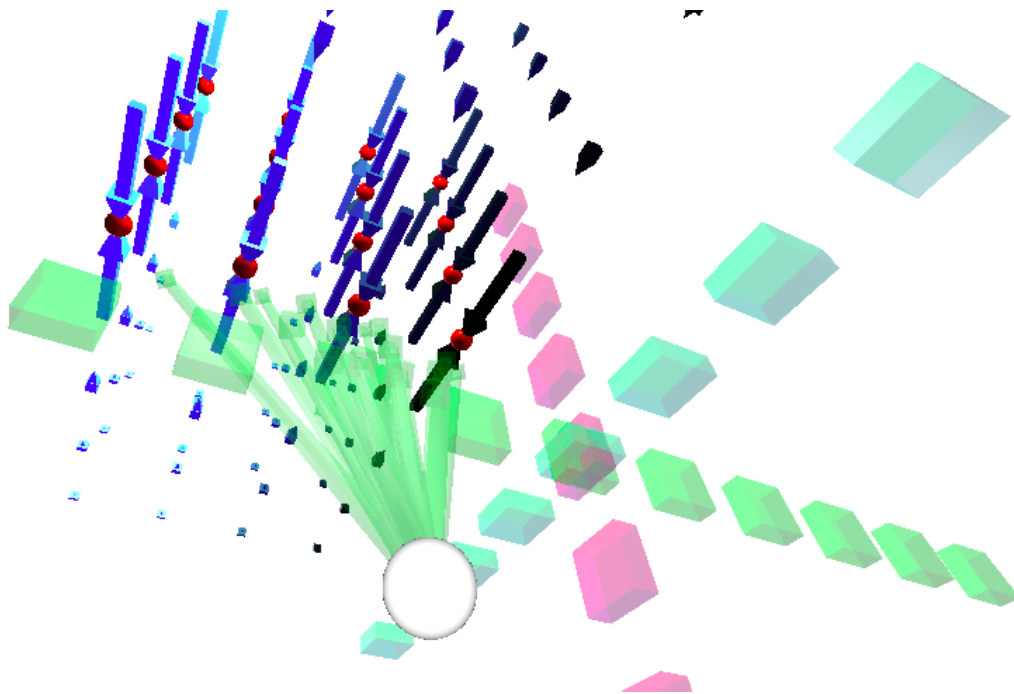
```

```
FieldVectors_TestCharge.append(arrow (pos = testCharge.pos, color =
↪ color.green, axis = 50*ScaleFactor*E_TestCharge[i], headwidth =
↪ sources[i].radius, shaftwidth = ((sources[i].radius)/2), opacity =
↪ 0.15))
```

Whenever we have such disparate parts as the vectors of an electric field, it's useful to combine them as a compound. To do so, we once again employ the `compound()` function to group these vectors together, thus making their manipulation trivial.

```
allEFVectors = []
for i in range(0,len(FieldVectors_TestCharge)):
    allEFVectors.append(FieldVectors_TestCharge[i])
    print("Positional Difference:" +
↪ (FieldVectors_TestCharge[i].pos-testCharge.pos))
greenElectricField = compound(allEFVectors)
```

Due to the microscopic scale at which the program runs, Glowscript makes rendering errors time to time. As such, minor corrections to the position of various elements may need to be applied – thus the need for positional checks such as the one above. We now consider the Net



Electric Field Vector at test charge Q . We could have simply pursued the traditional route and applied the $F_E = EQ = \frac{Q\sigma}{2\epsilon_0}$ for a sheet of flat charge, but the lackluster result would not have the same intuitive or visual impact as combining all the Electric Field Vectors at a point. Furthermore, it can be used for instructional purposes to demonstrate the Magnitude of the Electric Field for a flat sheet of charge remains constant by distance. With that motivation in mind, we now seek to find the net electric field at the test charge. To do so, we begin by summing the magnitudes and directions of the vectors separately.

```

for i in range (0,len(FieldVectors_TestCharge)-1):
    sumVectors = sumVectors + mag(FieldVectors_TestCharge[i].axis)
    dirVectors_x = dirVectors_x + FieldVectors_TestCharge[i].axis.x
    dirVectors_y = dirVectors_y + FieldVectors_TestCharge[i].axis.y
    dirVectors_z = dirVectors_z + FieldVectors_TestCharge[i].axis.z

```

With our net direction in each direction ready, we declare the direction of the net electric field vector to be

```

dirVectors = vector(dirVectors_x, dirVectors_y, dirVectors_z)

```

We proceed to adding the remaining vectors at Q , including the Electric Field, Magnetic Field, Electric Force, Magnetic Force, and velocity. The vectors are scaled accordingly to make them visible on the canvas.

```

netElectricFieldVector = arrow(pos = testCharge.pos, color = color.red,
    ↪ axis = 10000*ScaleFactor*(sumVectors)*norm(dirVectors), shaftwidth =
    ↪ sources[1].radius, opacity = 0.5)
netMagneticFieldVector = arrow(pos = testCharge.pos, color = color.cyan,
    ↪ axis = 1e23*ScaleFactor*(J*permeability_magnetic/2)*vector(0,0,1),
    ↪ shaftwidth = sources[1].radius, opacity = 0.6)
velocityVector = arrow(pos = testCharge.pos, color = color.orange, axis =
    ↪ vector(1e-8,0,0))
ElectricForce = arrow(pos = testCharge.pos, color = color.magenta, axis =
    ↪ .5*1e19*mag(netElectricFieldVector.axis)*(1.6e-19)*norm(EF.axis))
MagneticForce = arrow (pos = testCharge.pos, color = color.black, axis =
    ↪ -.6*10e26*Q*cross(netMagneticFieldVector.axis, velocityVector.axis))

```

Breakpoints such as the one below are occasionally scattered throughout the program to indicate the magnitude and direction of the above vectors.

```

print("The net electric field at Q is " + mag(netElectricFieldVector.axis)
    ↪ + "And the direction is " + norm(netElectricFieldVector.axis))

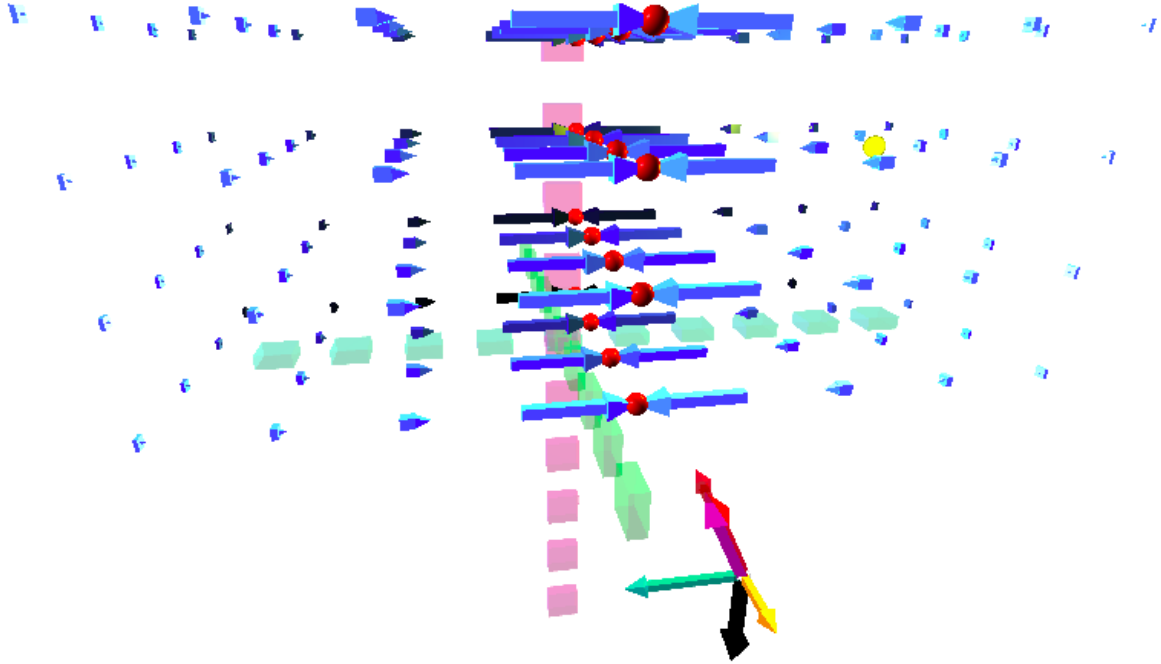
```

We now initialize various graphs that indicate the direction of motion, and the change in Electric Force (if any), among other computations. To construct a graph, we use the `graph()` function for all seven of our charts. For instance, creating the Magnitude of the Electric Field Vector by Distance would be

```

g1 = graph(title='Electric Field Vectors', scroll=False, fast=True, xmin =
    ↪ 19e-9, xmax = 20e-9, xtitle='Magnitude of Electric Field Vector',
    ↪ ytitle='Time')
f1 = gdots(graph=g1, color=color.red)

```



And likewise, we repeat for all seven graphs. We now increment the actual movement of the test Charge and the sheet. For elements that will move in a linear path, we will simply utilize a small time increment to animate said element's movement. In the case of the test charge, however, we have an explicitly defined equation $\vec{r}(t)$ that defines its movement. In addition, we also plot all seven of our graphs in the same `for` loop that animates the movement of our test Charge and Infinite Sheet, so that the graphs update in real-time. The excerpt has been modified for brevity, but the gist is to simply define the positional increment for all elements in the program, whilst simultaneously plotting the magnitude of different vectors according to distance, time, or other scalars.

```

for time in range (1,4):
    rate(0.2)
    secondMovingVectors.pos += deltaMove
    testCharge.pos.x = -1e2*(1e-10)*time
    testCharge.pos.y = 1e-22*(Q*E*(time**2))/(2*m)

    changeInPos = vector(2.5e-8,-2.5e-8,-2e-8) - testCharge.pos

    greenElectricField.opacity -= time
    netElectricFieldVector.pos = testCharge.pos

    f5.plot(mag(netElectricFieldVector.axis),time)

```

Finally, we conclude with a means of switching perspectives from the Stationary Observer to a Moving Observer (i.e., from solely an Electric Field to an Electric and Magnetic Field).

To do so, we utilize the `follow` attribute of the `camera` method for our scene. In addition, we add a Graphical User Interface for the user to switch between these modes and actually visually witness the charge moving faster as a moving observer rather than a stationary one. We may do so as follows:

```

pointer = sphere (radius = 1e-20, pos = vector(0,0,0))
scene2.camera.follow(testCharge)

def TestCharge():
    scene2.camera.follow(testCharge)
def Center():
    scene2.camera.follow(pointer)
def getDist():
    print("Camera Position "+scene.camera.pos)
    print("Camera Axis "+scene.camera.axis)

cbutton = button(text='Stationary Observer', pos=scene2.title_anchor,
    ↪ bind=Center)

```

And thus, we conclude with an actual, visually intuitive demonstration of the time relativistic effects of a simple test charge moving alongside an infinite sheet of charge. The pedagogical opportunities of this demonstration are evident – it enables the student to actually witness the relativistic behavior of time as applied to Electromagnetism, in stark contrast to simply deriving the Gamma Factor and Time Dilation formula, the traditional approach supplied by many textbooks.

5 $\frac{\partial \vec{E}}{\partial t}$ from Biot-Savart and STR

The traditional introduction of the Displacement Current term in standard electromagnetics textbooks is via an inconsistency in Ampere’s Law when applied to Parallel Plate Capacitors [26] – i.e., an Amperian Loop can be chosen such that a current does not penetrate the boundary of the surface. This presentation may lead one to believe the Displacement Current to be an *ad-hoc* correction to Ampere’s Law, when it is actually a natural consequence of a moving test charge [27] [28].

To motivate the existence of the displacement current, we chose an alternative route – not a parallel plate capacitor, but a simple test charge located at the origin.

According to Maxwell himself, the Displacement Current term is ‘electrostatically’ analogous to a normal current element or a moving test charge. We thus examine a test charge from a stationary and moving observer’s point of view. Whereas a stationary observer only finds an electric field, an observer moving with velocity $-v$ with respect to the stationary frame will witness both an electric and magnetic field. Such a test charge would thus exhibit the displacement current, due to the moving electric field. We take the relativistic form of the electromagnetic fields of the test charge and derive the Biot Savart Law from it. Hidden implicitly within the Biot Savart Law is the Displacement Current Term, which

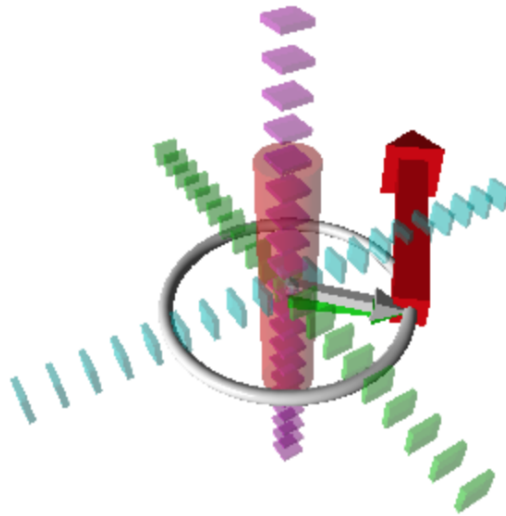
we reformulate using the Partial Derivative of a Cross Product to conclude with Ampere's Law, corrected with the Displacement Current Term. Before doing so, however, it is crucial to understand the Biot-Savart Law as it is traditionally applied to a current-carrying wire.

5.1 Visualization of Biot-Savart

We begin our demonstration of the Biot-Savart Law by initializing a scene frame and drawing the x, y, z axes as

```
scene = canvas(background = color.white)
for i in range (-10,10):
    rate(10)
    axesX = box(length = 0.2, width = 1, height = 1, pos = vector(i, 0,
        ↪ 0), color = color.green, opacity = 0.3)
    axesY = box(length = 1, width = 1, height = 0.2, pos = vector(0, i,
        ↪ 0), color = color.magenta, opacity = 0.3)
    axesZ = box(length = 1, width = 0.2, height = 1, pos = vector(0, 0,
        ↪ i), color = color.cyan, opacity = 0.3)
```

We immediately thereafter establish our current-carrying wire `chargedRod` and declare our drag state to be `False`. We create the wire using the `cylinder()` function in Glowscript



and initialize the `drag` state to `false` so that the student may be able to drag the position of the magnetic field vector in real-time.

```
chargedRod = cylinder (pos = vector(0,-5,0), axis = vector (0,10,0),
radius = 1, opacity = 0.3, color = color.red)
drag = False
```

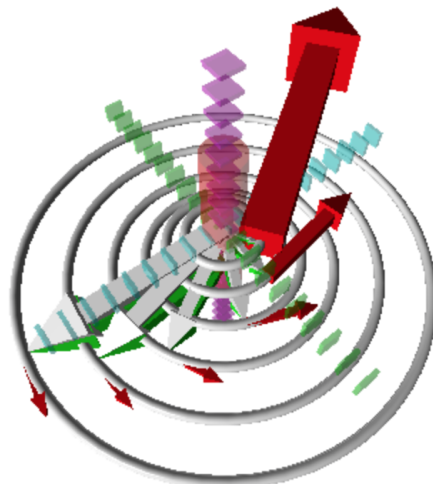

All relevant electromagnetic constants including μ_0 , I , and R are initialized. R depends on the distance of the position of the cursor from the origin.

```
mu_not = 4*pi*10e-7
R = scene.mouse.pos.y
biotSavart = (mu_not * I)/(2*pi*R)
```

We seek to create the magnetic field vector at any distance R from the current-carrying wire. To do so, we require both Magnitude and Direction. The magnitude will be supplied by the Biot-Savart Law as

$$\|\vec{B}\| = \frac{\mu_0 I}{2\pi R} \quad (93)$$

The Right Hand Rule gives the direction of the magnetic field to be counterclockwise, since the current has $\text{dir}(I) = \hat{j}$. The unexpected challenge, however, is to compute a consistently counterclockwise magnetic field \vec{B} vector around the wire. To do so, we construct three auxiliary vectors: a vector \vec{u} from the origin to the y axis, a vector \vec{p} from the origin to the cursor position, and a vector \vec{v} as the difference between the two vectors. From that difference, a dot product $\vec{v} \cdot \vec{w}$ is calculated such that the result is 0, and thus we have that \vec{w} is indeed the magnetic field vector at the cursor's position. Prior to constructing this in



code, we must declare three functions – each corresponding to three of the user's actions: impressing the track-pad, dragging, and releasing. We thus define our `down()` function to be

```
def down():
    drag = True
    origin = vector(0,0,0)
    originToMouse = arrow (pos = origin, axis = scene.mouse.pos-origin)
    originToZ = arrow (pos = origin, axis = vector(0,scene.mouse.pos.y,0))
    diffVec = arrow (pos = originToZ.axis, axis =
        ↪ originToMouse.axis-originToZ.axis, color = color.green)
```

```

fieldVector = arrow (pos = scene.mouse.pos, axis =
↳ scaleFactor*biotSavart*norm(vector(-scene.mouse.pos.z,0,
↳ scene.mouse.pos.x)))
circleShow = ring(radius=sqrt((scene.mouse.pos.x)**2+
↳ (scene.mouse.pos.z)**2), thickness = 0.25, pos =
↳ vector(0,scene.mouse.pos.y,0), axis = vector(0,1,0))

```

In the next auxiliary function `move()`, the program updates the magnitude and direction of all four vectors based on the position of the cursor. We must also declare a few global variables so they may be used outside the `move()` function. For instance, we may have

```

def move():
    global drag, fieldVector, originToMouse, originToZ, diffVec,
    ↳ circleShow
    if drag:
        originToMouse.axis = scene.mouse.pos - vector(0,0,0)
        circleShow.pos = vector(0,scene.mouse.pos.y,0)

```

Finally, we have the release function `up()`, which generates a graph of the magnitude of the magnetic field by distance and updates real-time, based on the user's placement of Magnetic Field vectors. We may verify the graph and indeed find that the magnetic field drops proportional to the distance R . Lastly, we bind the scene to all three auxiliary functions, leaving us with

```

def up():
    global drag, fieldVector, originToMouse, diffVec
    fieldVector.color = color.red
    fieldGraph = gdots(color=color.cyan, radius = 6, color = color.red)
    fieldGraph.plot(mag(diffVec.axis),mag(fieldVector.axis))
    drag = False
    scene.bind("mousedown", down)

```

5.2 Time Dilation from EM

We now pivot from the Biot-Savart Law to a moving test charge. In doing so, we will find that the Law we have just visually demonstrated implicitly contains within it the Displacement Current, without the need for any capacitors. Consider once again our test charge Q , centered at the origin and moving in the \hat{i} direction. As Maxwell himself states the Displacement Current to be "electrostatically" equivalent to a traditional current formed by a moving Electric Field, it makes sense to find the change in the Electric Field of the charge in respect to time. To do so, we employ the multivariable chain rule as

$$\frac{\partial E}{\partial t} = \frac{\partial E}{\partial x} \cdot \frac{\partial x}{\partial t} \quad (94)$$

$$\frac{\partial E}{\partial t} = (-v) \cdot \frac{\partial}{\partial x} \left(\frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \right) \quad (95)$$

Here's where relativity comes in. Through the Electromagnetic Field Strength Tensor, we find the equations of a relativistic electric field to be

$$E_{\parallel} = E'_{\parallel} \quad (96)$$

$$B_{\parallel} = 0 \quad (97)$$

$$E_{\perp} = \gamma E'_{\perp} \quad (98)$$

$$B = +\gamma \frac{1}{c_2} v \times E' \quad (99)$$

In which

$$\gamma = \frac{1}{\sqrt{1 - \frac{c^2}{v^2}}} \quad (100)$$

We thus have the Magnetic Field at any given distance R to be

$$B(r) = \gamma \frac{1}{c^2} v \times E = \gamma \mu_0 \frac{qv \times \hat{r}}{4\pi r^2} \quad (101)$$

There it is! The Displacement Current implicit within the Biot-Savart Law. By the partial derivative of a cross product, we have

$$B(r) = \gamma \frac{1}{c^2} (\vec{v} \times \vec{E}) \quad (102)$$

$$B(r) = \gamma \mu_0 \epsilon_0 (\vec{v} \times \vec{E}) \quad (103)$$

$$\frac{\partial}{\partial t} (\vec{r} \times \vec{E}) = \frac{\partial}{\partial t} (\vec{r}) \times \vec{E} + \vec{r} \times \frac{\partial}{\partial t} (\vec{E}) \quad (104)$$

$$\frac{\partial}{\partial t} (\vec{r} \times \vec{E}) = -\vec{v} \times \vec{E} + \vec{r} \times \frac{\partial}{\partial t} (\vec{E}) \quad (105)$$

$$0 = -\vec{v} \times \vec{E} + \vec{r} \times \frac{\partial}{\partial t} (\vec{E}) \quad (106)$$

$$\vec{r} \times \frac{\partial}{\partial t} (\vec{E}) = \vec{v} \times \vec{E} \quad (107)$$

$$B(r) = \gamma \mu_0 r \times \left(\epsilon_0 \frac{\partial E}{\partial t} \right) \quad (108)$$

And thus we conclude with none other than Ampere's Law, which includes Maxwell's addition of the Displacement Current:

$$\frac{\partial B}{\partial t} = \gamma \mu_0 v \times \left(\epsilon_0 \frac{\partial E}{\partial t} \right) \quad (109)$$

And therein lies the power of the Biot Savart Law – it implicitly contains the Displacement Current [19], without which the oscillation of the Electromagnetic Wave would remain a mystery.

6 Conclusion

Various extensions may be applied to the above computational programs to further increase their utility in demonstrating the bridge between Maxwell's Equations and Einstein's Theory of Special Relativity. For instance, for the Galilean Transformations visualization, `drag` states can be defined in the program to enable the student to change the velocities of the observers and position of the event E in real-time without having to re-run the program. The visualization for Time Dilation from an EM problem may be modified such that the student can define the charge's position and initial horizontal velocity. Furthermore, an analogous computational program may be created to demonstrate the radiation from a charge at rest in a gravitational field [25]. The final computational visualization on the Biot-Savart Law may be modified to enable the user to adjust various electromagnetic parameters, such as the current running through the wire.

Through the last three sections, we have demonstrated how computational programming visualizations can work symbiotically with theoretical derivations to solidify the concepts behind the bridge between EM and STR and make them tangible – indeed, computational visualizations such as the ones above offer rich pedagogical opportunities for teachers and students alike to intuitively grasp the ideas of Maxwell's Equations and Einstein's Special Relativity.

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