1 Sellmeier & Cauchy Equations

1. Starting with the general form of the dispersion equation according to the electron-oscillator model of the atom, derive the Sellmeier Equation (where A_j's are constants):

$$n^2(\lambda) = 1 + \Sigma_j A_j \frac{\lambda^2}{(\lambda^2 - \lambda_j^2)}$$

We begin with the refractive index's dependence on frequency

$$n^2(\omega) = 1 + \frac{Nq_e^2}{\epsilon_0 m_e} \sum_j \frac{f_j}{\omega_{0j}^2 - \omega^2 + i f_j \omega}$$

The damping term approaches 0 for $\omega_{0j} >> \omega$. Substituting $\omega = \frac{2\pi c}{\lambda}$, we have

$$n^2(\lambda) = 1 + \frac{Nq_e^2}{\epsilon_0 m_e} \sum_j \frac{f_j}{\left(\frac{2\pi c}{\lambda_{0j}}\right)^2 - \left(\frac{2\pi c}{\lambda}\right)^2}$$

Dividing the numerator and denominator by the common factor, we have

$$n^2(\lambda) = 1 + \sum_j A_j \cdot \frac{1}{\frac{4\pi^2 c^2}{\lambda_0^2} - \frac{4\pi^2 c^2}{\lambda^2}} \cdot \frac{\frac{1}{4\pi^2 c^2}}{\frac{1}{4\pi^2 c^2}}$$

If we pack all the constants into one term $A_j = \frac{Nq_e^2 f_j \lambda_{0j}^2}{\epsilon_0 m_e 4\pi^2 C^2}$, we have

$$\boxed{n^2(\lambda) = 1 + \sum_j A_j \frac{\lambda_j^2}{\lambda^2 - \lambda_{0j}^2}}$$

2. Show that the Cauchy Equation is an approximation of the Sellmeier Equation:

$$n(\lambda) = C_1 + \frac{C_2}{\lambda^2} + \frac{C_3}{\lambda^4} + ...$$

To show that the Sellmeier Equation is equivalent to Cauchy, we have to think several steps ahead.We'll try to pack the second term of $n^2(\lambda)$ into a simple taylor expansion, and then root it to get $n(\lambda)$, after which we may taylor expand again. We thus begin by getting the fraction term in the form $\frac{1}{1-x}$:

$$n^2(\lambda) = 1 + A\left(\frac{1}{1 - \frac{\lambda^2}{\lambda^2}}\right)$$

Recall the Taylor Expansion

$$\frac{1}{1-x} = 1 + x \text{ where } x = \frac{\lambda_0^2}{\lambda^2}$$

Substituting into $n^2(\lambda)$, we have

$$n^2(\lambda) = 1 + A\left(1 + \frac{\lambda_0^2}{\lambda^2}\right) = 1 + A + A\frac{\lambda_0^2}{\lambda^2}$$

Cauchy is $n(\lambda)$, which leads us to recall the taylor expansion for the square root:

$$\begin{array}{lll} f(x) = (1+x)^{\frac{1}{2}} & f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}} & f''(x) = -\frac{1}{4}(1+x)^{-\frac{3}{2}} \\ f(0) = 1 & f'(0) = \frac{1}{2} & f''(0) = -\frac{1}{4} \end{array}$$

$$f(x) \approx \frac{f(0)}{0!}x^0 + \frac{f'(0)}{1!}x' + \frac{f''(0)}{2!}x^2 = 1 + \frac{1}{2}x - \frac{1}{8}x^2 \pm \dots$$

We may now exploit this Taylor expansion to give

$$n(\lambda) = \sqrt{1 + A\left(1 + \frac{\lambda_0^2}{\lambda^2}\right)}, x = A\left(1 + \frac{\lambda_0^2}{\lambda^2}\right)$$

$$n(\lambda) = 1 + \frac{1}{2}x = 1 + \frac{1}{2}\left[A\left(1 + \frac{\lambda_0^2}{\lambda^2}\right)\right]$$

We can now see the direct correspondence between the Sellmeier and Cauchy Equation:

$$\boxed{n(\lambda) = 1 + \frac{1}{2}A + \frac{1}{2}A\frac{\lambda_0^2}{\lambda^2}, C_1 = 1 + \frac{1}{2}A, C_2 = A\frac{\lambda_0^2}{2}}$$

3. Crystal quartz has refractive indices of 1.557 and 1.547 at wavelengths of 410 nm and 550 nm, respectively. Using only the first two terms of the Cauchy's Equation, calculate C_1 and C_2 and find the index of refraction of quartz at 610 nm.

   Substituting the known values, we have

$$n(410 \text{ nm}) = C_1 + C_2/(410 \text{ nm})^2 = 1.557$$
$$n(550 \text{ mm}) = C_1 + C_2/(550 \text{ nm})^2 = 1.547$$

We will now use the Cauchy Equation to create a system of equations from which we may find C_1 and C_2.

$$\boxed{C_1 = 1.5345, C_2 = 3783.501}$$

We may now leverage this new information to find the refractive index at 610 nm:

$$n(610 \text{ nm}) = 1.5345 + 3783.501/(610 \text{ nm})^2 \rightarrow \boxed{n(610 \text{ nm}) = 1.544}$$

2 Absorption Coefficient

1. The absorption coefficient is $\alpha(\omega) = 2\omega\kappa/c$. Show that for a dilute medium and near resonance, the absorption coefficient is

$$\alpha(\omega) = \frac{Ne^2}{4m\varepsilon_0 C}\frac{\gamma}{(\omega_0 - \omega)^2 + (Y/2)^2}$$

Let us begin by justifying why the absorption coefficient should be defined as $\alpha(\omega) = 2\omega\kappa/c$ in the first place. We begin by defining a complex version of the wavenumber k:

$$\tilde{k} = k + i\kappa$$

Why do we define a complex wavenumber? It turns out that doing so gives

$$E(z,t) = E_{0z}e^{i(\tilde{k}z-\omega t)} = E_{0z}e^{i(kz+ikz-\omega t)} = E_{0z}e^{-kx}e^{i(kz-\omega t)}$$

We can clearly see that there is an attenuation factor of e^{-kz} which exponentially decays the wave. This should make sense, as we expect the damping term to cause an effect once the wave enters the dielectric. Indeed, since $I \propto E^2$ (since $I = \frac{1}{2}\epsilon_0 v E^2$), it is reasonable to define

$$a \equiv 2\kappa$$

The goal now is to get κ. We can do so by first establishing a relationship between the refraction index n and $\tilde{k}$.

$$n = \frac{c}{v} \text{ and } v = \frac{\omega}{k} = \frac{2\pi b}{\frac{2\pi}{\lambda}} = b\lambda \rightarrow n = \frac{c\tilde{k}}{\omega}$$

Recall that the refractive index n may also be written as

$$n = \frac{c}{\nu} = \sqrt{\frac{\epsilon}{\epsilon_0}\frac{\mu}{\mu_0}} = \sqrt{\tilde{\epsilon}_r\tilde{\mu}_r}$$

But $\mu = \mu_0$ for dielectrics, which gives

$$n = \sqrt{\tilde{\epsilon}_r} \rightarrow \tilde{k} = \frac{\omega}{c}\sqrt{\tilde{\epsilon}_r}$$

Recall that the complex relative permittivity is given as[1]

$$\tilde{\epsilon} = 1 + \frac{Nq_e^2}{m_e\epsilon_0}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}$$

Recall the taylor expansion $\sqrt{1+x} \approx 1 + \frac{1}{2}x$, where $x = \frac{Nq_e^2}{m_e\epsilon_0}\sum_j \frac{f_j}{\omega_j^2-\omega^2-i\gamma_j\omega}$:

$$\tilde{\kappa} = \frac{\omega}{c}\left(1 + \frac{Nq_e^2}{2m\epsilon_0}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - iy_j\omega}\right)$$

[1]This will be derived in Problem 4

But remember the whole motivation behind this calculation was to find the absorption term, which is linked to the imaginary component of the wavenumber $\tilde{\kappa}$. With this in mind, it is useful to remember that

$$\frac{1}{a-bi} \cdot \frac{a+bi}{a+bi} = \frac{a+bi}{a^2+b^2} = \frac{a}{a^2+b^2} + i\frac{b}{a^2+b^2}$$

We thus multiply the numerator and denominator by the complex conjugate, which gives

$$\tilde{k} = \frac{\omega}{c}\left(1 + \frac{Nq_e^2}{2m\epsilon_0}\sum_j \frac{f_j}{\left(\omega_j^2-\omega^2\right) - (i\gamma_j\omega)} \cdot \frac{\left(\omega_j^2-\omega^2\right) + (i\gamma_j\omega)}{\left(\omega_j^2-\omega^2\right) + (i\gamma_j\omega)}\right)$$

Simplifying this expression and distributing the complex conjugate gives

$$\tilde{k} = \frac{\omega}{c}\left(1 + \frac{Nq_e^2}{2m\epsilon_0}\sum_j \frac{f_j\left(\omega_j^2-\omega^2\right) + f_j(i\gamma_j\omega)}{\left(\omega_j^2-\omega^2\right)^2 + (\gamma_j\omega)^2}\right)$$

Isolating the real and imaginary components, we have

$$\tilde{k} = \frac{\omega}{c}\left(1 + \frac{Nq_e^2}{2m\epsilon_0}\left[\sum_j \frac{f_j\left(\omega_j^2-\omega^2\right)}{\left(\omega_j^2-\omega^2\right)^2 + (\gamma_j\omega)^2} + i\sum_j \frac{f_j\gamma_j\omega}{\left(\omega_j^2-\omega^2\right)^2 + (\gamma_j\omega)^2}\right]\right)$$

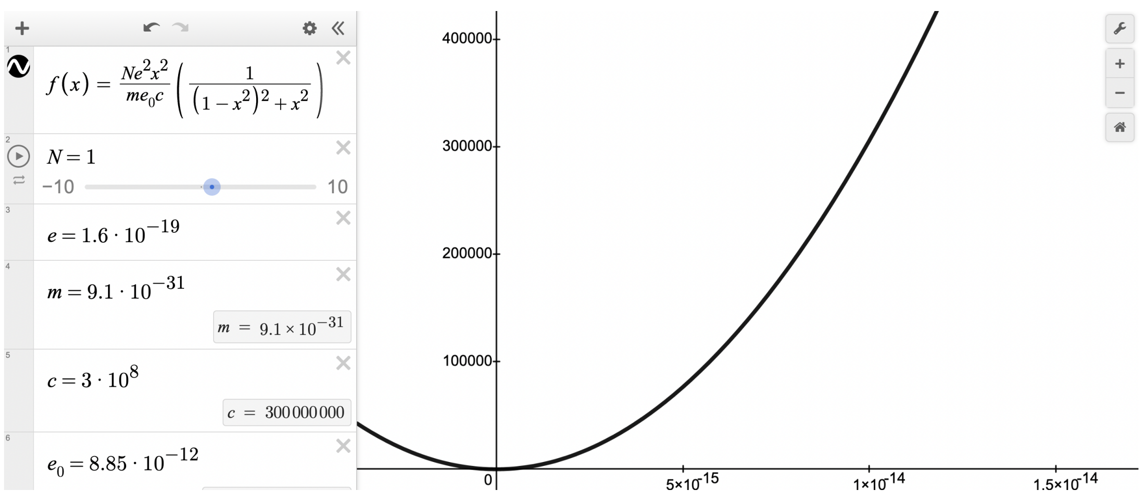

Figure 1: Plot of Absorption Coefficient

Solely the imaginary component is of interest, as we seek the absorption term

$$\kappa = \frac{\omega}{c}\left(\frac{Nq_e^2}{2m\epsilon_0}\sum_j \frac{f_j\gamma_j\omega}{\left(\omega_j^2-\omega^2\right)^2 + (\gamma_j\omega)^2}\right)$$

As $\alpha = 2\kappa$, we have

$$\boxed{\frac{Ne^2\omega^2}{m\epsilon_0 c}\sum_j \frac{f_j\gamma_j}{\left(\omega_j^2-\omega^2\right)^2+\left(\gamma_j\omega\right)^2}}$$

3 Light Intensity

1. How far does a 1.55μm beam travel before dropping 10% in intensity?

Using relation between intensity and distance, we have

$$I(y) = I_0 e^{-\alpha y} \to .90 I_0 = I_0 e^{-\alpha y}$$

Solving for y by taking the natural log of both sides, we have

$$y = \frac{-\ln(.9)}{\alpha} = \frac{-c\ln(.9)}{2\omega k}$$

We know all variables in this expression, except ω, which we may substitute with

$$f = \frac{\nu}{\lambda} \to f \cdot 2\pi = \frac{2\pi\nu}{\lambda} \to \omega = 2\pi\frac{\nu}{\lambda} \to \omega = \frac{2\pi c}{n\lambda}$$

We may now substitute the givens:

$$y = \frac{-cn\lambda\ln(.9)}{4\pi c\kappa} = \frac{-(1.5)(1.55\mu\text{m})\ln(.9)}{4\pi\left(3\times10^{-8}\right)} \to \boxed{y = 0.649\text{m}}$$

4 Lorentz Oscillator Model

1. How is an atom modeled?

Lorentz's motivation may have been to model the anomalous dispersion regions of the $n(\omega)$ plot by likening them to the effects of a simple harmonic oscillator when driven at resonant frequency ω_0. A damped oscillator may be expressed as

$$F_{\text{binding}} + F_{\text{damping}} + F_{\text{restoring}} = \sum F$$

Substituting for the terms and leveraging $\omega_0 = \sqrt{\frac{k}{m}} \Rightarrow k = \omega_0^2 m_e$, we have

$$q_e E_0\cos(\omega t) - m_e\gamma\frac{dx}{dt} - kx = m_e\frac{d^2x}{dt^2}$$

Though it may seem arbitrary, expressing the position function as a complex $\tilde{x}$ and expressing $\vec{E}(t)$ as an exponential will confer calculation benefits when solving for $\tilde{x}$. It is also worth noting that the electron will likely oscillate at the driving frequency, hence the ω.

$$\tilde{x}(t) = \tilde{x}_0 e^{-i\omega t} \to \frac{d\tilde{x}}{dt} = -i\omega\tilde{x}_0 e^{-i\omega t} \to \frac{d^2\tilde{x}}{dt^2} = i^2\omega\tilde{x}_0 e^{-i\omega t}$$

Shifting all terms involving $\tilde{x}$ to the left side and dividing by m_e, we have

$$q_e E_0 \cos(\omega t) - \omega_0^2 m_e \tilde{x} - m_e \gamma \frac{d\tilde{x}}{dt} = m_e \frac{d^2\tilde{x}}{dt^2}$$

$$\omega_0^2 \tilde{x} + y \frac{d\tilde{x}}{dt} + \frac{d^2\tilde{x}}{dt^2} = \frac{q_e}{m_e} \underbrace{E_0 \cos(\omega_0 t)}_{E_0 e^{-i\omega t}}$$

We may now substitute the derivatives of $\tilde{x}$:

$$\omega_0^2 \tilde{x} + \gamma \left(-i\omega \tilde{x}_0 e^{-i\omega t}\right) + \left(-\omega^2 \tilde{x}_0 e^{-i\omega t}\right) = \frac{q_e}{m_e} E_0 e^{-i\omega t}$$

Noting that $\frac{d\tilde{x}}{dt} = -i\omega\tilde{x}$ and $\frac{d\tilde{x}}{dt} = -i\omega\tilde{x}$ simplifies our expression significantly:

$$\omega_0^2 \tilde{x} + \gamma(-i\omega\tilde{x}) + \left(-\omega^2 \tilde{x}\right) = \frac{q_e}{m_e} E_0 e^{-i\omega t}$$

All that remains is to solve for $\tilde{x}$:

$$\tilde{\chi} \left[\omega_0^2 - \omega^2 - i\gamma\omega\right] = \frac{q_e}{m_e} E_0 e^{-i\omega t}.$$

$$\boxed{\tilde{\chi} = \frac{q_e/m_e}{\omega_0^2 - \omega^2 - i\gamma\omega} E_0 e^{-i\omega t}}$$

2. How do the natural frequencies relate to the dispersion curve for a material?

What we just derived is the position function of an electron in a dielectric when driven by an oscillating electric field. However, we may relate this to the refractive index by linking it to the total dipole moment P in the medium, which involves the charges q in the medium, the position function x of the charges, and the number of charges N.

$$P = N q_e x$$

We may subsequently link the dipole moment to the permittivity ϵ,as such:

$$\epsilon = \epsilon_0 + \frac{P(t)}{E(t)}$$

Then it's just a matter of algebra to link ϵ to the refractive index n, since $n \approx \sqrt{\epsilon_r}$ for dielectrics. The subsequent expression for $n^2(\omega)$ is exactly the equation which gives rise to the dispersion curve. And indeed, when the natural frequency ω_0 is equal to the driving frequency ω, we note that the refractive index shoots up, with the damping term $i\gamma_j\omega$ as its only saving grace. Therein lies the relationship between the natural frequencies and the dispersion curve of the material.

3. What is meant by normal dispersion? Why is it so called?

Normal dispersion are the regions of the $n(\omega)$ plot in which the refractive index is seen to rise with increasing frequency. This is reasonable, as we see blue light ($750 * 10^{12}$ Hz) – which is of higher frequency than red light ($430 * 10^{12}$ Hz) – being bent significantly greater when exiting a prism. Normal dispersion is so called because it is what is often observed in the index of refraction vs. frequency plot.

4. What is anomalous dispersion? Why is it so called? What actually happens in those regions?

   Anomalous dispersion are regions where the refractive index n is seen to shoot up or plummet seemingly arbitrarily, at the resonant frequencies of the dielectric. It is called anomalous because it is not often observed, and is seen only when $\omega \approx \omega_0$. If we neglect the damping term, we see that $\omega = \omega_0$ is a vertical asymptote for the refractive index $n^2(\omega)$, which explains why we see an abrupt jump or fall in the refractive index.

5 Derivation of Fresnel's Equations

1. Starting with Fresnel's Equations, derive the corresponding expressions in terms of trigonometric functions only.

   I will not start with Fresnel's Equations. Instead, I will begin at the very beginning, by deriving the two boundary conditions needed to derive all four Fresnel's Equations. I will then use Snell's Law to simplify the four equations to their requested form above.

   Each of Maxwell's four equations gives rise to a boundary condition, but we will need only two – Ampere's Law and Faraday's Law. We begin with Ampere:

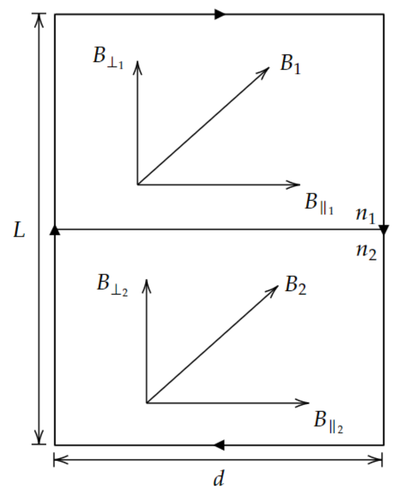

Figure 2: Amperian Loop constructed around two Mediums

   One is justified in asking why we are employing Ampere's Law in the first place[2] to derive boundary conditions. The answer is subtle, but important: to model the interface between the two mediums, we may use exactly what Maxwell offers us: a Gaussian surface of nil height or an Amperian Loop of zero width! Therein lies the key idea behind using Maxwell's equations to derive the boundary conditions.

   By Ampere's Law, we have

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$

[2]Or any of the Maxwell's Equations, for that matter

Note that the vertical components $B_{\perp_1}$ and $B_{\perp_2}$ do not contribute to the line integral on the top and bottom, but do indeed contribute to the line integral on the left and right sides. However, the contribution that $B_{\perp_1}$, for instance, makes on the left side line integral, is exactly the same (but in opposing sign) to the contribution it makes on the right side line integral. It thus cancels out its own contributions. Likewise for $B_{\perp_2}$. On the other hand, the horizontal components $B_{\parallel_1}$ and $B_{\parallel_2}$ do not contribute to the side line integrals, but do indeed contribute to the top and bottom line integrals, and they survive!

$$B_{\parallel_1} d - B_{\parallel_2} d = \mu_0 I + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$

But behold! As I shrink my Amperian Loop, letting $L \to 0$, as was my intention all along, I am left with no electric flux or current, since there is no closed loop to speak of. We thus have

$$B_{\parallel_1} d - B_{\parallel_2} d = 0 \to \boxed{B_{\parallel_1} = B_{\parallel_2}}$$

And this is our first boundary condition. To obtain our second boundary condition, we use an almost identical strategy, but with Faraday's Law for the electric field $\vec{E}$, instead of Ampere for $\vec{B}$.

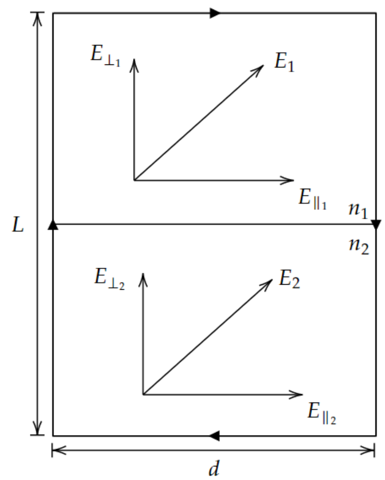

Figure 3: Amperian Loop constructed around two Mediums

By Faraday's Law, we have

$$\oint \vec{E} \cdot \overrightarrow{ds} = -\frac{d\Phi_B}{dt}$$

Once again, only the horizontal components $E_{\parallel_1}$ and $E_{\parallel_2}$ make a contribution, and as we let $L \to 0$, we have

$$\boxed{E_{\parallel_1} = E_{\parallel_2}}$$

Behold, our second boundary condition. I will now proceed to derive all four of Fresnel's equations, which will subsequently be simplified using Snell's Law. At this point, it is critically important to understand the geometry of the situation. I was confused for a long time, for instance, why the tangential components transmitting meant that $E_{0i} + E_{0r} = E_{0t}$ or why we take the $\cos(\theta)$ of $\vec{B}$ when $\vec{E}$ is perpendicular to the plane of incidence, and $\cos(\theta)$ of $\vec{E}$ when $\vec{B}$ is perpendicular to the plane of incidence. Let me begin by addressing these two confusions.

First of all, why does the tangential components transmitting imply that $E_{0i}+E_{0r} = E_{0t}$? To answer that question, it is helpful to consider an analogous scenario, in which we have springs instead of electromagnetic waves.

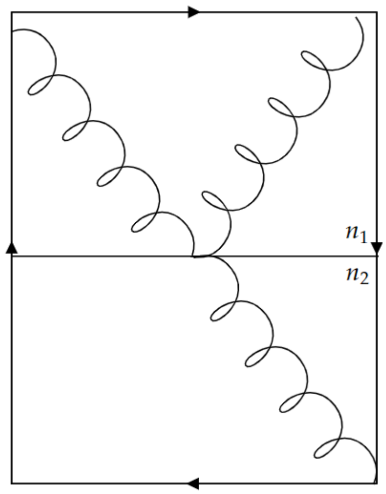

Figure 4: Three Springs connected in Series

Would it not be reasonable to claim that at the junction point between the upper springs and lower spring, the amplitude of all the springs must match? But wait – this would be equivalent to requiring equal phase terms amongst E_{0i}, E_{0r}, and E_{0t}. But if the phase terms are equivalent and the amplitudes must match, that leaves us with only one option – that the sum of the amplitudes of the two upper spring must equal to the amplitude of the lower spring at the junction point! In other words, $\vec{E}_{0i} + \vec{E}_{0r} = \vec{E}_{0t}$!

I will now address my second confusion: why was $\cos(\theta)$ of $\vec{E}$ taken when it was parallel, but not perpendicular to the plane of incidence (and likewise for $\vec{B}$)? The answer, unsurprisingly, lies in the geometry of the situation. There is a very subtle thing going on, and that is this: when the electric field vector $\vec{E}$ is perpendicular to the plane of incidence, it's projection is *itself*. Think about it for a moment, and you'll realize why this is true. However, if you start removing the plane of the incoming electric field even a bit, you've got to use $\cos(\theta)$ to bring its projection to the boundary and pass it through. This is a very key idea that relies critically on geometric intuition.

Having addressed my two major confusions for Fresnel's Equations, I now proceed to consider the case in which $\vec{E}$ is perpendicular to the plane of incidence. This results in

$$\vec{E}_{0i} + \vec{E}_{0r} = \vec{E}_{0t}$$

Since $\vec{B}$ is in the plane, it's projection is not itself, and we thus have

$$-B_i \cos(\theta_i) + B_r \cos(\theta_r) = -B_t \cos(\theta_t)$$

The negative signs are due to the opposing directions of B_i and B_t. See diagram below.

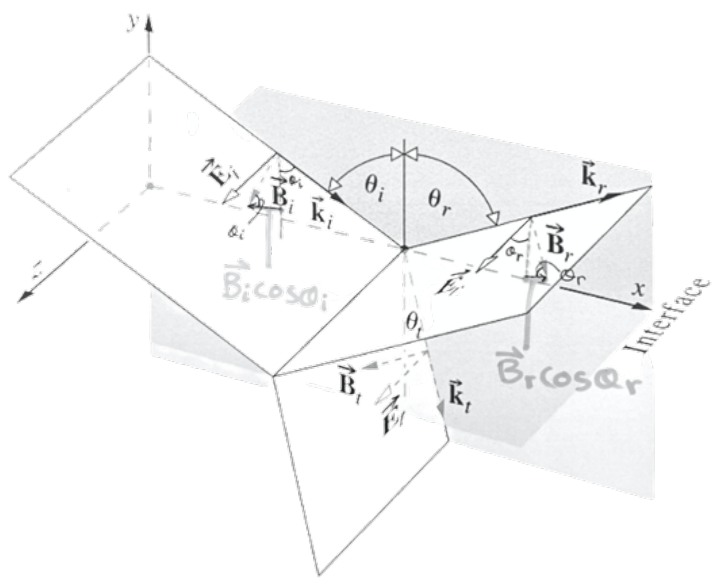

Figure 5: Hand-Annotated version of Hecht Diagram (Resolution to Confusion 2)

To know what step to take next, it is worthwhile to remind ourselves of the ultimate goal: to find the intensity of the reflected and transmitted beams. That is the ultimate utility of Fresnel's equations. But to find the intensity, we must obtain a ratio of the electric fields first. To do so, we realize that we must convert the above equation involving $\vec{B}$ into one involving $\vec{E}$. How so? Consider the relation

$$E = vB \rightarrow B = \frac{E}{v}$$

This relationship emerges from $\vec{k} \times \vec{E} = -i\omega\vec{B}$, which is a consequence of Faraday's Law. Substituting this relation into our equation, we have

$$-\frac{1}{v_i} E_{0i} \cos(\theta_i) + \frac{1}{v_r} E_{0r} \cos(\theta_r) = -\frac{1}{v_t} E_{0t} \cos(\theta_t)$$

We recognize that as we are dealing with two dielectrics of different refractive indices, n must somehow be involved in the final expression. We thus note an opportunity to introduce n by multiplying both sides by c:

$$-n_i E_{0i} \cos(\theta_i) + n_r E_{0r} \cos(\theta_r) = -n_t E_{0t} \cos(\theta_t)$$

But $n_i = n_r$ (same medium) and $\theta_i = \theta_r$ by the law of reflection!

$$-n_i E_{0i} \cos(\theta_i) + n_r E_{0r} \cos(\theta_r) = -n_t E_{0t} \cos(\theta_t)$$

$$n_i \left(E_{0i} - E_{or}\right) \cos\theta_i = n_t E_{0t} \cos\theta_t$$

Substituting $\vec{E}_{0i} + \vec{E}_{0r} = \vec{E}_{0t}$, we have (disregarding μ, as it is ≈ 1 for dielectrics)

$$\begin{aligned} &\tfrac{n_i}{\mu_i} E_{0i} \cos\theta_i - \tfrac{n_t}{\mu_t} E_{0i} \cos\theta_t = \tfrac{n_i}{\mu_i} E_{0r} \cos\theta_i + \tfrac{n_t}{\mu_t} E_0 \cos\theta_t \\ &E_{0i} \left(\tfrac{n_i}{\mu_i} \cos\theta_i - \tfrac{n_t}{\mu_t} \cos\theta_t\right) = E_{\text{or}} \left(\tfrac{n_i}{\mu_i} \cos\theta_i + \tfrac{n_t}{\mu_t} \cos\theta_t\right) \end{aligned}$$

It is now a matter of mere algebra, which ultimately leads to

$$\boxed{r_\perp = \left(\frac{E_{0r}}{E_{0i}}\right)_\perp = \frac{n_i \cos\theta_i - n_t \cos\theta_t}{n_i \cos\theta_i + n_t \cos\theta_t}}$$

This is our first Fresnel equation, the amplitude reflection coefficient! To get the transmission coefficient, we simply express $\vec{E}_{0r}$ in terms of $\vec{E}_{0t}$:

$$\vec{E}_{0r} = \vec{E}_{0t} - \vec{E}_{0i}$$

This gives our second Fresnel equation, the amplitude transmission coefficient!

$$\boxed{t_\perp \equiv \left(\frac{E_{0t}}{E_{0i}}\right)_\perp = \frac{2 n_i \cos\theta_i}{n_i \cos\theta_i + n_t \cos\theta_t}}$$

We are well on our way to our third Fresnel Equation, which begins much the same way as our first two, but with one caveat: this time, the electric field $\vec{E}$ is parallel to the plane of incidence, and $\vec{B}$ is perpendicular to it. How the tables have turned! We must now do to $\vec{E}$ what we once did to $\vec{B}$: project it onto the plane, since it's projection is no longer itself. This gives

$$E_{0i} \cos\theta_i - E_{\text{or}} \cos\theta_r = E_{0t} \cos\theta_t$$

And of course, this time there is no need to project $\vec{B}$, as its tangential component happily transmits across the interface.

$$\vec{B}_{0i} + \vec{B}_{0r} = \vec{B}_{0t}$$

We follow the same line of thought: introducing n where possible and simplifying the expression, which gives

$$\frac{1}{v_i} E_{0i} + \frac{1}{v_r} E_{0r} = \frac{1}{v_t} E_{ot}$$

$$n_i E_{oi} + n_r E_{or} = n_t E_{0t}$$

Leveraging the equality $n_i = n_r$, we have

$$\cos(\theta_i)(E_{0i} - E_{0r}) = \cos(\theta_t) E_{0t}$$

But hang on – $E_{0t} = \frac{n_i}{n_t}(E_{0i} + E_{0r})\cos(\theta_t)$, which gives

$$E_{oi} \cos\theta_i - E_0 \cos\theta_r = \frac{n_i}{n_t}(E_i + E_{0r})\cos\theta_t$$

$$n_t \cos(\theta_i)(E_{0i} - E_{0r}) = n_i (E_{0i} + E_{0r})\cos(\theta_t)$$

This results in our third Fresnel equation!

$$\boxed{r_\parallel = \frac{n_t \cos\theta_i - n_i \cos\theta_t}{n_i \cos\theta_t + n_t \cos\theta_i}}$$

To find it's transmission counterpart, we simply make the substitution

$$E_{0i} \cos\theta_i - \left(\frac{n_t}{n_i} E_{0t} - E_{0_i}\right)\cos\theta_r = E_{0t}\cos\theta_t$$

The rest is basic algebra, solving for the ratio of the electric fields:

$$2E_{0i}\cos\theta_i - \frac{n_t}{n_i} E_{0t}\cos\theta_r = E_{0t}\cos\theta_t$$

$$2E_{0i}\cos\theta_i = E_{0t}\left(\frac{n_t}{n_i}\cos\theta_r + \cos\theta_t\right)$$

$$\frac{E_{\text{ot}}}{E_{0i}} = \frac{2\cos\theta_i}{\frac{n_t}{n_i}\cos\theta_r + \cos\theta_t}$$

One last simplification, and we have the transmission coefficient for an electric field parallel to the plane of incidence!

$$\boxed{t_\parallel = \left(\frac{E_{0t}}{E_{0i}}\right)_\parallel = \frac{2n_i\cos(\theta_i)}{n_t\cos(\theta_i) + n_i\cos(\theta_t)}}$$

Now that we have our four Fresnel's equations, let's simplify them using Snell's Law. We begin with the reflection coefficient for $\vec{E}$ perpendicular to the plane of incidence:

$$r_\perp = \left(\frac{E_{or}}{E_{oi}}\right)_\perp = \frac{n_i\cos\theta_i - n_i\frac{\sin\theta_i}{\sin\theta_t}}{\eta_i\cos\theta_i + \eta_i\frac{\sin\theta_i}{\sin\theta_t}}$$

$$r_\perp = \left(\frac{E_{or}}{E_{oi}}\right)_\perp = -\frac{\sin(\theta_i)\cos(\theta_t) - \cos(\theta_i)\sin(\theta_t)}{\sin(\theta_i)\cos(\theta_t) + \cos(\theta_i)\sin(\theta_t)}$$

Notice that this is nothing but the sum and difference of sines:

$$\boxed{r_\perp = -\frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)}}$$

We now proceed to repeat this for the transmission coefficient for the electric field perpendicular to the plane of incidence. Since

$$n_i \sin\theta_i = n_t \sin\theta_t \rightarrow n_t = n_i \frac{\sin\theta_i}{\sin\theta_t}$$

We can make the substitution into the transmission coefficient, which gives

$$t_\perp = \frac{2n_i \cos\theta_i}{n_i \cos\theta_i + \left(n_i \frac{\sin\theta_i}{\sin\theta_t}\right)\cos\theta_t}$$

Multiplying the numerator and denominator by $\sin(\theta_t)$, we have

$$t_\perp = \frac{2n_i \sin\theta_t \cos\theta_i}{n_i \cos\theta_i \sin\theta_t + n_i \cos\theta_t \sin\theta_i}$$

A final algebraic simplification reduces the denominator to a sum of sines, which gives

$$t_\perp = \frac{2n_i \sin\theta_t \cos\theta_i}{n_i\left(\sin\left(\theta_i + \theta_t\right)\right)} \rightarrow \boxed{t_\perp = \frac{2\sin\theta_t \cos\theta_i}{\sin\left(\theta_i + \theta_t\right)}}$$

We now consider the reflection coefficient for the parallel case, which proceeds similarly:

$$r_{11} = \frac{\left(n_i \frac{\sin\theta_i}{\sin\theta_t}\right)\cos\theta_i - n_i \cos\theta_t}{n_i \cos\theta_t + \left(n_i \frac{\sin\theta_i}{\sin\theta_t}\right)\cos\theta_i}$$

Multiplying through by $\sin(\theta_t)$, we have

$$r_{11} = \frac{n_i \sin\theta_i \cos\theta_i - n_i \cos\theta_t \sin\theta_t}{n_i \cos\theta_t \sin\theta_t + n_i \sin\theta_i \cos\theta_i}$$

A final rearranging of the terms and cancellation of n_i reveals a striking similarity between the numerator and denominator:

$$\boxed{r_\parallel = \frac{\sin\theta_i \cos\theta_i - \cos\theta_t \sin\theta_t}{\sin\theta_i \cos\theta_i + \cos\theta_t \sin\theta_t}}$$

And now, for our final fresnel simplification, we have

$$t_\parallel = \frac{2n_i \cos\theta_i}{n_i \cos\theta_t + \left(\frac{n_i \sin\theta_i}{\sin\theta_t}\right)\cos\theta_i}$$

Multiplying through by $\sin(\theta_t)$, we have

$$t_\parallel = \frac{2n_i \sin\theta_t \cos\theta_i}{n_i \sin\theta_t \cos\theta_t + n_i \sin\theta_i \cos\theta_i}$$

Finally, from the ashes, rises our transmission coefficient, simplified using Snell's Law:

$$\boxed{t_\parallel = \frac{2 \sin\theta_t \cos\theta_i}{\sin\theta_t \cos\theta_t + \sin\theta_i \cos\theta_i}}$$

I have thus derived the two boundary conditions, derived all four of Fresnel's Equations, and simplified all four using Snell's Law.

6 Transmitted and Reflected Intensity

1. Find the intensity of the reflected beam and that of the transmitted beam independently.

   I struggled with this question for a long time because I wasn't sure how to handle the unpolarized nature of natural light. My first idea was to decompose the electric field vector $\vec{E}$ into a vector half-polarized parallel and half-polarized perpendicular to the field, and likewise for $\vec{B}$, but I wasn't sure how to apply that to find the actually intensities.

2. Find the angle of incidence for which the reflected light will be completely plane polarized.

   This is a simple application of Brewster's angle, which requires $\theta_p = \frac{\pi}{2} - \theta_t$, which gives

$$\tan(\theta_p) = \frac{n_2}{n_1} = \frac{1.52}{1.33} \to \boxed{49.0001^\circ}$$

7 Near Normal Incidence

1. Show that at near-normal incidence, the reflection coefficient may be approximated as

$$[-r_\perp]_{\theta i} \simeq 0 = [(n-1)/(n+1)] \times \left[1 + \left(\theta_i^2\right)/n\right]$$

Our first order of business will be to "normalize" the indices of refraction by considering $n_i = 1$ and $n_t = n$, which significantly simplifies our expression to

$$r_\perp = \frac{\cos\theta_i - n\cos\theta_t}{\cos\theta_i + n\cos\theta_t}$$

Similar to the case for $r_\parallel$, we now leverage the taylor expansion of $\cos(\theta)$:

$$\cos(\theta) \approx 1 - \frac{\theta^2}{2}$$

Which may be substituted into $r_\perp$ to give

$$r_\perp = \frac{\left(1 - \frac{\theta_i^2}{2}\right) - n\left(1 - \frac{\theta_t^2}{2}\right)}{\left(1 - \frac{\theta_i^2}{2}\right) + n\left(1 - \frac{\theta_t^2}{2}\right)}$$

We now leverage Snell's Law, combined with the small-angle approximation for θ, which gives $\theta_t = \frac{\theta_i}{n}$:

$$r_\perp = \frac{\left(1 - \frac{\theta_i^2}{2}\right) - n\left(1 - \frac{\theta_i^2}{2n^2}\right)}{\left(1 - \frac{\theta_i^2}{2}\right) + n\left(1 - \frac{\theta_i^2}{2n^2}\right)} = \frac{1 - \frac{\theta_i^2}{2} - n + n\frac{\theta_i^2}{2n^2}}{1 - \frac{\theta_i^2}{2} + n - n\frac{\theta_i^2}{2n^2}}$$

Factoring out common terms, we have

$$r_\perp = \frac{(1-n) + \frac{\theta_i^2}{2}\left(\frac{1}{n} - 1\right)}{(1+n) + \frac{\theta_i^2}{2}\left(-\frac{1}{n} - 1\right)}$$

Noting a common $(1-n)$ and $(1+n)$, we have

$$r_\perp = \frac{(1-n)\left[1 + \frac{\theta_i^2}{2n}\right]}{(1+n)\left[1 - \frac{\theta_i^2}{2n}\right]}$$

We may now taylor expand $\frac{1}{1-x}$ as $1+x$, which gives

$$r_\perp = \frac{1-n}{1+n}\left(1 + \frac{\theta^2}{2n}\right)\left(1 + \frac{\theta^2}{2n}\right) = \frac{1-n}{1+n}\left(1 + \frac{\theta^2}{2n}\right)^2$$

Expanding the binomial term, we conclude with our desired result

$$-r_\perp = \frac{n-1}{n+1}\left(1 + \frac{\theta_i^2}{n} + O(\theta_i^4)\right) \to \boxed{(-r_\perp)_{\theta_i \approx 0} = \frac{n-1}{n+1}\left(1 + \frac{\theta_i^2}{n}\right)}$$

8 Parallel-Polarized Reflection Coefficient

1. Starting with the relevant Fresnel equation, obtain an expression for $r_\parallel$ in terms of the angle of incidence and relative index of refraction $n_{ti} = \frac{n_t}{n_i}$. Show that for total internal reflection, $r_\parallel$ is a complex quantity and $R_\parallel = 1$.

   We begin by considering $r_\parallel$ as is, and dividing through by n_i, which gives

$$r_\parallel = \frac{n_t \cos\theta_r - n_i \cos\theta_t}{n_t \cos\theta_r + n_i \cos\theta_t} \cdot \frac{\frac{1}{n_i}}{\frac{1}{n_i}} = \frac{n_{ti}\cos\theta_r - \cos\theta_t}{n_{ti}\cos\theta_r + \cos\theta_t}$$

   Since $n_i \sin(\theta_i) = n_t \to n_{ti} = \sin(\theta_i)$, we have

$$r_\parallel = \frac{\sin\theta_r \cos\theta_r - \cos\theta_t}{\sin\theta_r \cos\theta_r + \cos\theta_t}$$

   Simplifying further, we have

$$r_\parallel = \frac{\frac{1}{2}\sin(2\theta_i) - \cos\theta_t}{\frac{1}{2}\operatorname{tin}(2\theta_i) + \cos\theta_t}$$

Our desired result is thus

$$\boxed{r_{\parallel} = \frac{\sin(2\theta_i) - 2\cos\theta t}{\sin(2\theta_i) + 2\cos\theta_t}}$$

For total internal reflection, we have $\theta_t = 90°$, which results in $\cos(\theta_t) = 1$, which gives

$$\boxed{R = r *_{\parallel} r_{\parallel} = 1}$$

9 Snell's Cone

1. A fish looking straight up at a smooth surface of a pond sees a circular field surrounded by darkness. Explain what is happening. If the fish is at a depth D from the surface and the refraction index of the water is n, find the cone-angle and area of the circular field. Find the angle of the cone-angle.

   This is a most interesting phenomenon, which I suspect occurs due to something involving the critical angle. If the fish is looking straight up vertically, he will of course see what is directly above him, but as the angle between the light rays emerging from his fish eyes and the surface of the pond gradually increases, at some point, the refracted beam will hit the critical angle, at which point there will be no refraction, and utter darkness for the fish. Furthermore, since light is moving from a denser to a lighter medium (water to air), that makes it all the more likely that the light rays are diverging further from the normal, and hitting the critical angle at an earlier point, resulting in a darkness surrounding the circular field of view. As the critical angle from water to air is $1.33\sin(\theta_i) = \sin(0) \rightarrow \theta_i = 48.75$, the cone angle should be $\frac{\pi}{2} - 48.75 = \boxed{41.24°}$. The area of the cone is elementary to deduce, as we now have the angle of the cone (with respect to the surface normal) and the fish's distance D from the surface. The radius of the circular field of view should thus be $D\sin(\theta) = D\sin(41.24)$, which would give an area of $A = \pi(D\sin(41.24))^2$.