1 State of Polarization

1. Describe completely the state of polarization of each of the following waves:

$$\mathbf{E}(z,t) = \mathbf{i}E_0\cos(kz-\omega t) + \mathbf{j}E_0\cos(kz-\omega t+\pi/2)$$

$$\mathbf{E}(z,t) = \mathbf{i}E_0\sin(\omega t-kz) + \mathbf{j}E_0\sin(\omega t-kz-\pi/4)$$

We begin with the first wave. The wave cannot be linearly polarized, since the phase terms are different. It must therefore be circularly polarized, since the amplitudes are equal ($|E_x| = |E_y|$). We must now identify the handedness of the polarization. We can do so in two different ways: first, we may recognize that E_y leads E_x by $\frac{\pi}{2}$ and they have equal amplitudes. Thus, the wave is R, right circularly polarized. Alternatively, we may also consider the wave at different points in time for a fixed value of position:

$$E(0,t) = \mathbf{i}E_0\cos(\omega t) + \mathbf{j}E_0\cos(\omega t-\pi/2)$$

$$E\left(0,\frac{T}{8}\right) = \mathbf{i}E_0\cos(\frac{\pi}{4}) + \mathbf{j}E_0\cos\left(\frac{\pi}{4}-\pi/2\right)$$

$$E\left(0,0\right) = \mathbf{i}E_0$$

$$E\left(0,\frac{T}{4}\right) = \mathbf{j}E_0\cos\left(\frac{\pi}{2}-\pi/2\right)$$

This confirms the first wave is indeed right circularly polarized. We similarly conclude that for the second wave, E_y lags E_x by $\frac{\pi}{4}$ and they have equal amplitudes. We thus have a L, a left circularly polarized wave for the second case.

2 Polarization & Brewster's Angle

1. A beam of natural light reflected from a glass ($n_g = 1.65$) plate immersed in ethyl alcohol ($n_e = 1.36$) is found to be completely linearly polarized. At what angle will the partially polarized beam will be transmitted into the plate? Determine the degree of polarization of the transmitted beam.

We begin by employing Snell's Law to find θ_t:

$$n_i\sin(\theta_i) = n_t\sin(\theta_t) \rightarrow \theta_t = \sin^{-1}\left(\frac{n_i}{n_t}\sin\theta_i\right) \rightarrow \boxed{\sin^{-1}\left(\frac{1.65}{1.36}\sin 51.1^\circ\right) \rightarrow \theta_t = 70.8^\circ}$$

To determine the degree of polarization of the transmitted beam, we look to Fresnel's Equations, which I derived in the previous homework problem set. Specifically, we look at $T_\parallel$ and $T_\perp$:

$$T_\parallel = \left(\frac{2\sin\theta_t\cos\theta_i}{\sin(\theta_i+\theta_t)}\right)^2 = \left(\frac{2\sin(70.8)\cos(51.1)}{\sin(51.1+70.8)}\right)^2 = 1.951785069$$

$$T_\perp = \left(\frac{2\sin\theta_t\cos\theta_i}{\sin(\theta_i+\theta_t)\cos(\theta_i-\theta_t)}\right)^2 = \left(\frac{2\sin(70.8)\cos(51.1)}{\sin(51.1+70.8)\cos(51.1-70.8)}\right)^2 = 2.202006098$$

We further recall that the overall transmittance is given by

$$T = \frac{1}{2}(T_{\parallel} + T_{\perp}) = \frac{1}{2}(1.951785069 + 2.202006098) = 2.076895584$$

To find the degree of polarization, we would finally use

$$V = \frac{I_P}{I_P + I_N} = \frac{T_{\parallel} + T_{\perp}}{T_{\parallel} + T_{\perp} + T} = \frac{4.153791167}{2.076895584} = \boxed{\frac{2}{3} \approx 0.6667}$$

3 Malus' Law

1. An enthusiastic Physics 20900 student wants to rotate the plane of polarization of a polarized light beam by a total of 45° using a sequence of ideal polarizing filters. The polarizing axis of each filter makes the same angle with that of the previous filter in the sequence. The student wants an intensity reduction of no more than 10%. How many polarizers does the student need? What is the angle between the polarizing axes of two adjacent polarizers?

   My idea is simple: use Malus' Law repeatedly for each polarizer, since the incoming light is already polarized (otherwise we would simply halve the intensity). We seek

$$0.90 I_0 = I_0(\cos(\theta))^2$$

   But we are not using one polarizer – we're doing as many as it takes to rotate the plane by 45° while not reducing the intensity more than 10%. The angle θ is thus $\theta = \frac{45}{n}$. For n polarizers,

$$0.90 = \left(\cos\left(\frac{45}{n}\right)\right)^{2n}$$

   It seems $\boxed{n = 6}$ polarizers gives the desired result with a final intensity of 90.2% and an angle of $\boxed{\theta = \frac{45}{6} = 7.5^\circ}$ between each polarizer.

4 More Malus!

1. A beam of natural light with intensity $50\frac{W}{m^2}$ is sent into a system of two polarizing sheets P1 and P2. The polarizing axis of P1 makes an angle $\theta_1 = 50^\circ$ with +y-axis (40° with +x-axis) and the polarizing axis of P2 makes an angle $\theta_2 = 20^\circ$ with +y-axis (110° with +x-axis). What is the intensity and polarization of the light transmitted by the two-sheet system, assuming the light is first incident on P1? How will the situation change if the light were incident on P2 and came out through P1?

   This is a case of simply applying Malus' Law to the system. Since the light is unpolarized (i.e., natural), its intensity is halved as soon as it passes through the first polarizer. Passing through the second polarizer leaves $I_0(\cos(30))^2$, which is $\frac{3}{4}$ of the remaining intensity. That means the final outcoming light has an intensity $I_f = \frac{3}{4}\frac{1}{2}I_0 = \boxed{0.375I_0}$, and it is polarized according to the polarization of the second filter P_2, which is 20° to the +y-axis. If the light were incident on P_2 instead, it would once again lose half its intensity after the first polarizer

P_2. By the time it passed through the second polarizer, it would have lost the same intensity, because the angular difference in the polarization axis for P_1 and P_2 is the same: still 30°. The only difference is that the final polarization of the light will be that of P_1 instead of P_2: 50° with respect to the +y-axis.

2. Find the intensity and polarization of the transmitted light if the beam were polarized parallel to +x axis and incident on P1 and coming out of P2, and vice versa.

   The great thing about Malus' Law is that the actual angle of the polarization axes do not matter – it is the relative angles that matter. Thus, the incident light will still lose half its intensity, but this time, since it was originally polarized along the +x-axis, it will lose an additional $(\cos(70°))^2$ of intensity, since $110° - 40° = 70°$. This means $\boxed{I_f = \frac{1}{5}(\cos(70°))^2 I_0 = 0.058 I_0}$. The polarization would match the polarization axis of the final polarizer, which would be P_1, which is polarized 40° with respect to the +x-axis. If the light had been incident upon P_2 and came out of P_1 instead, the final intensity would remain the same (since the relative angular difference in the polarization axes is the same), but the polarization would now be 110° with respect to the +x-axis.

5 Internal and External Reflection

1. Show that the polarization angles for internal and external reflection at a given interface are complementary, that is, $\theta_p + \theta'_p = \frac{\pi}{2}$.

   Recall that Brewster's Angle (the angle needed for polarization by reflection) is given by

$$n_i \sin(\theta_p) = n_t \sin(90 - \theta_p) \rightarrow n_i \sin(\theta_p) = n_t \cos(\theta_p) \rightarrow \tan(\theta_p) = \frac{n_t}{n_i}$$

In external reflection, light travels from a medium of lower index to one of higher index (i.e., vacuum to water). Internal reflection considers the opposite (i.e., $n_i > n_t$). We thus find that

$$\theta_p + \theta'_p = \arctan(x) + \arctan\left(\frac{1}{x}\right)$$

Proving that this quantity is equivalent to $\frac{\pi}{2}$ is trivial once we consider a right triangle with side lengths x and 1. We thus have $\tan(a) = x$ and $\tan(b) = \frac{1}{x}$. Since a and b are complementary, we have

$$\boxed{\theta_p + \theta'_p = \arctan(x) + \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2}}$$

6 Critical Angles

1. Find the polarizing angle for a 589 nm beam of light when it is incident from vacuum on a quartz (index of refraction, 1.46) slab. What will the polarizing angle be when the beam is traveling in quartz and is incident on the quartz-vacuum interface? Under which of the above two cases do you expect to see total internal reflection? What is the critical angle?

Recall that Brewster's angle is given by

$$\theta_p = \arctan\left(\frac{n_t}{n_i}\right) = \arctan(1.46) = \boxed{55.59^\circ}$$

When the beam of light is traveling quartz to vacuum, we have $\theta_p' = \frac{\pi}{2} - \theta_p = 90 - 55.59 = \boxed{34.41^\circ}$, as desired. We expect to observe total internal reflection (TIR) when light travels from a high refractive index medium to a low refractive index medium, since it will diverge even further away from the normal, making it easier to reach $\theta_t = 90^\circ$. I thus expect to see TIR in the case of light traveling from quart to vacuum. The critical angle is given by

$$n_i \sin(\theta_i) = n_t \sin(90) \to \theta_i = \sin^{-1}\left(\frac{n_t}{n_i}\right) = \sin^{-1}\left(\frac{1}{1.46}\right) = \boxed{43.23^\circ}$$

7 Half Wave & Quarter Wave Plates

1. The refractive indices of the uniaxial birefringent crystal sodium nitrate at the vacuum wavelength of 589.3 nm are $n_o = 1.5854$ and $n_e = 1.3369$. Find the minimum thickness of the crystal that will be needed to make (a) a half wave plate, and (b) a quarter wave plate.

   Employing the relations between the thickness of a plate, the refractive indices and the wavelength of light, we have

$$t_{1/4} = \frac{\lambda}{4(n_o - n_e)} = \frac{589.3 * 10^{-9}}{4(1.5854 - 1.3369)} = \boxed{592.85 * 10^{-9} m}$$

   Likewise for the half-wave plate, we have

$$t_{1/2} = \frac{\lambda}{2(n_o - n_e)} = 2t_{1/4} = \boxed{1185.714 * 10^{-9} m}$$