# 1 Michelson-Morley Interferometer

In one of the early (1881) versions of the Michelson-Morley experiment, the interferometer arms were 50 cm long. What would be the expected shift $\Delta N$ when the apparatus is rotated through $90°$, assuming that 590-nm light was used, and that the expected speed of the earth relative to the ether was $3 * 10^4$ m/s? Comment on the efficacy of this attempt of the experiment. Why is the apparatus rotated through $90°$ in this experiment?

## 1.1 Solution

The fringe shift is given by

$$\Delta N = \frac{L_1 + L_2}{\lambda}\frac{v^2}{c^2} = \frac{50 * 10^{-2}m + 50 * 10^{-2}m}{590 * 10^{-9}m}\frac{(3 * 10^4 m/s)^2}{(3 * 10^8 m/s)^2} = \boxed{0.01694915254}$$

We thus expect a shift of about one-hundredth of a fringe. The experiment is highly effective in measuring phase shifts, due to Michelson's ingenious interforemeter setup, where the fringe shift would result in a different interference pattern at the detector screen, and the setup was capable of detecting up to fringe shifts as small as $\frac{1}{100}$. The apparatus was rotated $90°$ to change the phase difference between the interferometer two paths, and this phase difference would in turn cause a fringe shift, which would enable Michelson and Morley to calculate the velocity $v$ of the ether.

## 1.2 Derivation

Let's derive the formula for phase shift $\Delta N = \frac{l_1+l_2}{\lambda}\frac{v^2}{c^2}$. Remember, the goal of the Michelson-Morley experiment was to experimentally confirm the existence of the "aether", the postulated medium that light was believed to travel in. To this end, Michelson had to create some measurable effect out of the aether. The genius of their experiment was to create a setup in which light would split and travel two different paths, and the interference pattern on the detector would reveal whether the light beams had arrived at the detector at the same time or had been retarded by the ether.

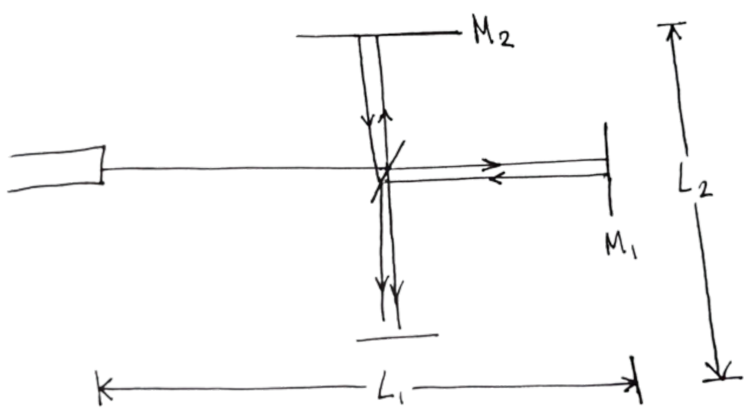


Figure 1: Interforemeter Setup

A simple analogy which may make it easier to understand the experiment goes as follows: two boats of velocity $c$ make round trips across and along a river with a current $v$. By switching the boats so that the boat traveling horizontally is now traveling vertically and vice-versa, we may calculate the time difference between the original setup and the current setup. This will tell us the velocity of the current!

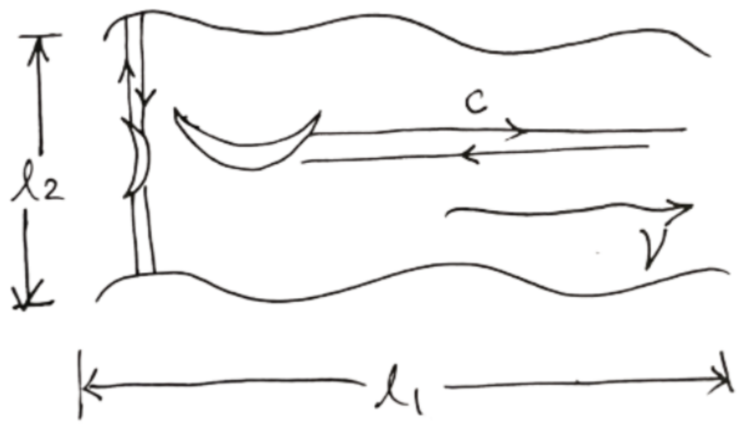


Figure 2: Boat Analogy of Michelson-Morley Experiment

This is in essence Michelson's genius idea: to calculate the velocity of the ether by rotating the entire apparatus and measuring the fringe shift on the interference pattern! A fringe shift would imply that light was impeded by the ether in one direction and assisted by the ether in another! Here is the setup of the experiment: it involves a beam splitter, two mirrors, $M_1$ and $M_2$, and a detector.

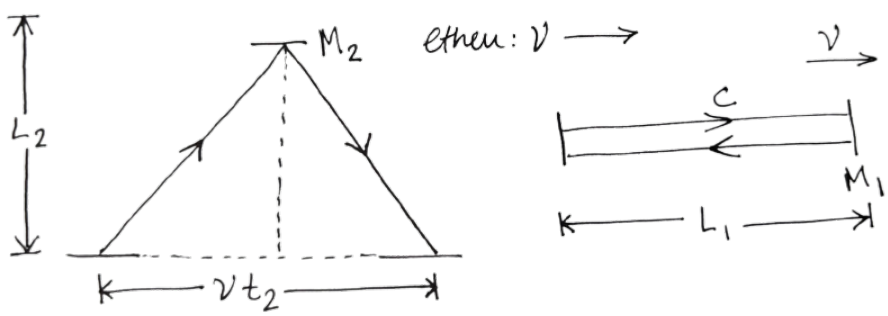


Figure 3: Isolated Light Beams along $L_1$ and $L_2$ from Ground Frame

Coherent, monochromatic light travels through the beam splitter and splits into a light beam traveling along $L_1$ and a beam along $L_2$. The beam along $L_1$ reflects off of $M_1$ and returns to the beam splitter, where it is reflected and travels into the detector. Likewise for the vertical beam, which travels to $M_2$, reflects and transmits through the beam splitter, traveling to the detector. If the two light beams arrive in-phase, there should be constructive interference. Otherwise, there will be destructive interference. Since the path lengths are the same, we expect no phase shift.

But, consider this! If the ether is, say, moving with a velocity $v$ to the right, it will increase the velocity of the light beam along $L_1$ on the forward journey (i.e., $c+v$), but retard its velocity on the return trip (i.e., $c-v$). On the other hand, since we presume the ether to move horizontally with velocity $v$, it doesn't affect the velocity of the light beam on the vertical path, but it does change the path length! In the frame of the earth, the ether is traveling at $v$, which makes it seem as if $M_2$ is actually moving and the light beam is traveling in a *diagonal* path, trying to catch up to it! This is just like watching a moving light clock from the ground!

We now proceed to find the times $t_1$ and $t_2$ along lengths $L_1$ and $L_2$:

$$t_1 = \frac{L_1}{c+v} + \frac{L_1}{c-v} = \frac{L_1(c-v)+L_1(c+v)}{(c+v)(c-v)} = \frac{L_1c - L_1v + L_1c + L_1v}{c^2-v^2} = \frac{2L_1}{c}\left(\frac{1}{1-v^2/c^2}\right)$$

$$t_2 = \frac{2\sqrt{\left(\frac{vt_2}{2}\right)^2 + L_2^2}}{c} \to ct_2 = 2\sqrt{\left(\frac{vt_2}{2}\right)^2 + L_2^2} \to c^2t_2^2 = 4\left[\frac{v^2t_2^2}{4} + L_2^2\right] \to c^2t_2^2 = v^2t_2^2 + 4L_2^2$$

$$t_2^2(c^2-v^2) = 4L_2^2 \to t_2^2 = \frac{4L_2^2}{c^2-v^2} \to t_2 = \frac{2L_2}{\sqrt{c^2-v^2}} = \frac{2L_2}{c\sqrt{1-c^2/v^2}} = \frac{2L_2}{c}\left(\frac{1}{\sqrt{1-v^2/c^2}}\right)$$

The time difference along the paths is then

$$\Delta t = t_2 - t_1 = \frac{2}{c}\left[\frac{L_2}{\sqrt{1-v^2/c^2}} - \frac{L_1}{1-v^2/c^2}\right]$$

If we switched the paths so that $L_1$ was perpendicular to the ether and $L_2$ parallel, we'd simply switch out the denominators:

$$\Delta t' = t_2' - t_1' = \frac{2}{c}\left[\frac{L_2}{1-v^2/c^2} - \frac{L_1}{\sqrt{1-v^2/c^2}}\right]$$

The time difference after we rotate is thus

$$\Delta t' - \Delta t = \frac{2}{c}\left[\frac{L_2}{1-v^2/c^2} - \frac{L_1}{\sqrt{1-v^2/c^2}} - \frac{L_2}{\sqrt{1-v^2/c^2}} + \frac{L_1}{1-v^2/c^2}\right]$$

$$\Delta t' - \Delta t = \frac{2}{c}\left[\frac{L_1+L_2}{1-v^2/c^2} - \frac{L_1+L_2}{\sqrt{1-v^2/c^2}}\right] = \frac{2}{c}(L_1+L_2)\left[\frac{1}{1-v^2/c^2} - \frac{1}{\sqrt{1-v^2/c^2}}\right]$$

Recall that $(1+m)^n \approx 1+mn$, which gives

$$\Delta t' - \Delta t = \frac{2}{c}(L_1+L_2)[1+v^2/c^2 - (1+\frac{1}{2}v^2/c^2)] = \frac{2}{c}(L_1+L_2)[\frac{1}{2}v^2/c^2] = \frac{L_1+L_2}{c}\frac{v^2}{c^2}$$

If there is a one wavelength path difference, there will be a shift of one fringe. Since we would like to get the wavelength of the light involved in the equation (as that is a known, measurable quantity), we have

$$\Delta N = \frac{c(\Delta t' - \Delta t)}{\lambda}$$

Check that this makes sense. $c(\Delta t' - \Delta t)$ is the extra path length difference after the apparatus is rotated. If that is equal to one wavelength, then that cancels out with the denominator to produce a shift of one fringe. We thus have

$$\Delta N = \frac{c(\Delta t' - \Delta t)}{\lambda} = \frac{c}{\lambda}\left(\frac{L_1 + L_2}{c}\frac{v^2}{c^2}\right) \rightarrow \boxed{\Delta N = \frac{L_1 + L_2}{\lambda}\frac{v^2}{c^2}}$$

# 2 Pion Decay

Pions have a half-life of 18 ns. A pion beam leaves an accelerator at a speed of 0.8c. (a) If relativistic effects are not considered, what is the distance over which half of the pions in the beam are expected to decay? (b) Taking relativistic effect into consideration, how far an observer in the laboratory will see the beam travel before approximately half of the pions in the beam decay? What distance will an observer riding on the pion beam will measure? How are the results consistent? The half-life is measured in a frame in which the particles are at rest.

## 2.1 Solution

The nonrelativistic distance traveled by the pion is $d = rt = (18*10^{-9}s)(0.8*3*10^8 m/s) = \boxed{4.32m}$. In the rest frame of the frame, the Pion lives

$$(18 * 10^{-9}s) * \frac{1}{\sqrt{1 - (.8c/c)^2}} = 30 * 10^{-9}s = 30ns$$

which translates to $30ns * 0.8c = \boxed{7.2m}$. An observer riding on the pion will measure

$$7.2m * \frac{1}{\sqrt{1 - (.8c/c)^2}} = 12m$$

This is sensible, as the observer on the pion experiences length contraction. The results are consistent because we can transform between the frame of the observer on the pion and the observer on the ground by employing the Lorentz Transformations.

# 3 Proper Time

An observer determines that two events are separated by $3.6 * 10^8$ m and occur 2s apart. What is the proper time interval between the occurrences of these two events?

### 3.1 Solution

The space-time interval will always provide the proper time or length for an object. The proper time is the shortest possible time measured between two events, because it is the time measured by the observer experiencing both events on their worldline. We have

$$s = \sqrt{c^2 \Delta t^2 - \Delta x^2} = \sqrt{2^2 - \left(\frac{3.6 * 10^8 m}{3 * 10^8 m/s}\right)} = \boxed{1.6s}$$

Thus, the proper time between the two events is 1.6 seconds, which is smaller than the time measured by the observer, as expected.

# 4 Relativistic Velocity Addition

A particle moves with a speed 0.8c at an angle of 30° with x-axis as measured by an observer in frame $O$. What is the velocity of the particle as measured by a second observer in frame $O'$ that is moving with speed $-0.6$c relative to $O$ along the common $x - x'$ axis?

### 4.1 Solution

We denote the particle's velocity in the $O$ frame $u$ and in the $O'$ frame $u'$. Let's now decompose the particle's velocity in $O$:

$$u_x = (0.8)\cos 30 = 0.6928c, u_y = (0.8)\sin 30 = 0.4c$$

We know that relativistic velocity transforms as

$$u'_x = \frac{u_x - v}{1 - (u_x v)/c^2} = \frac{(0.6928c) - (-0.6c)}{1 - ((0.6928c)(-0.6c))/c^2} = 2.73962164 * 10^8 m/s$$

$$u'_y = \frac{u_y \sqrt{1 - v^2/c^2}}{1 - (u_x v)/c^2} = \frac{(0.4c)\sqrt{1 - (0.4c)^2/c^2}}{1 - (((0.6928c))(-0.6c))/c^2} = 0.6781135081 * 10^8 m/s$$

The total velocity in $O'$ is thus

$$u' = \sqrt{u'^2_x + u'^2_y} = \sqrt{(2.73962164 * 10^8 m/s)^2 + (0.6781135081 * 10^8 m/s)^2} = \boxed{2.822297763 * 10^8 m/s}$$

We expect the velocity $u'$ in $O'$ to be greater than the velocity $u$ in $O$, since $O'$ is moving away from $O$. Indeed, $2.822 * 10^8 m/s > 0.8c$.

# 5 Fizeau Experiment

The speed of light in still water is $c/n$, where the index of refraction of water is approximately $n = 4/3$. In 1851, Fizeau found that the speed of light in water moving with speed $V$ relative to the laboratory may be expressed as $u = c/n + kV$ and he measured the "dragging coefficient" to be $k \approx 0.44$. Find the relativistic prediction for the value of $k$.

## 5.1 Solution

We employ the relativistic addition of velocities with $u_x = \frac{c}{n}$:

$$u'_x = \frac{u_x + v}{1 + (u_x v)/c^2} = \frac{\frac{c}{n} + v}{1 + \frac{\frac{c}{n}v}{c^2}} = \frac{\frac{c}{n} + v}{1 + \frac{v}{cn}} \to V - \frac{c}{n} = \frac{\frac{c}{n} + v}{1 + \frac{v}{cn}} - \frac{c}{n} = \frac{v(1 - \frac{1}{n^2})}{1 + \frac{v}{cn}}$$

For $\frac{v}{c} << 1$, we may reduce our expression to

$$v(1 - \frac{1}{n^2}) \to k = 1 - \frac{1}{n^2} = 1 - \frac{1}{(4/3)^2} = \boxed{0.4375}$$

# 6 Relativistic Energy

An electron is moving with velocity $5 * 10^7$ m/s. How much energy is needed to double the speed?

## 6.1 Solution

The energy for the electron is $E = \sqrt{m_e^2 c^4 + p^2 c^2}$, where $p_i = mv\gamma$ and $p_f = 2mv\gamma$. This gives $\Delta E = \sqrt{m_e^2 c^4 + (2mv\gamma_f)^2 c^2} - \sqrt{m_e^2 c^4 + (mv\gamma_i)^2 c^2}$. We know $v_i = 5 * 10^7 m/s$ and $v_f = 10^8 m/s$ and the lorentz gamma factors are

$$\gamma_i = \frac{1}{\sqrt{1 - (5 * 10^7/c)^2}} = 1.014185106, \gamma_f = \frac{1}{\sqrt{1 - (10^8/c)^2}} = 1.060660172$$

We now substitute all values to find

$$E_i = \sqrt{(9.109 * 10^{-31} kg)^2 (3 * 10^8 m/s)^4 + (9.109 * 10^{-31} kg)^2 (5 * 10^7 m/s)^2 (1.014)^2 (3 * 10^8 m/s)^2} = 8.314{*}10^{-14} J$$

$$E_f = \sqrt{(9.109 * 10^{-31} kg)^2 (3 * 10^8 m/s)^4 + 2(9.109 * 10^{-31} kg)^2 (5 * 10^7 m/s)^2 (1.060)^2 (3 * 10^8 m/s)^2} = 8.325{*}10^{-14} J$$

The difference in energy is thus

$$\Delta E = E_f - E_i = 8.325 * 10^{-14} J - 8.314 * 10^{-14} J = \boxed{1.0819 * 10^{-16} J}$$

# 7 Relativistic Momentum

The rest energy of a particle is 5 GeV, and its kinetic energy is found to be 8 GeV. What is momentum (in GeV/c) and what is its speed?

## 7.1 Solution

To find the relativistic momentum $p = mv\gamma$, we need the mass and velocity of the particle. We may find the mass from the rest energy:

$$E_{rest} = mc^2 = 5GeV \to m = \frac{5}{c^2} = 5.55 * 10^{-17} GeV/c^2$$

We may calculate the velocity from the relativistic kinetic energy:

$$E_{kinetic} = mc^2(\gamma - 1) = 8GeV \rightarrow \gamma = \frac{8}{mc^2} + 1 = \frac{8}{(5.55 * 10^{-17} GeV/c^2)(3 * 10^8 m/s)^2} + 1 = 2.6$$

We can extract the velocity from $\gamma$:

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \rightarrow \sqrt{1 - v^2/c^2} = \frac{1}{\gamma} \rightarrow 1 - v^2/c^2 = \frac{1}{\gamma^2} \rightarrow v = \sqrt{c^2\left(1 - \frac{1}{\gamma^2}\right)}$$

$$v = \sqrt{c^2\left(1 - \frac{1}{2.6^2}\right)} = \boxed{2.76923 * 10^8 m/s}$$

The momentum of the particle is thus

$$p = mv\gamma = (5.55 * 10^{-17} GeV/c^2)(\frac{2.76923 * 10^8 m/s}{3 * 10^8 m/s})(2.6) = \boxed{1.333 * 10^{-16} GeV/c}$$

# 8 Rest Energy

Find the rest energy of a proton and a neutron in MeV. Use rest mass of proton, $m_p = 1.672622 * 10^{-27}$ kg and that of the neutron $m_n = 1.674927 * 10^{-27}$ kg.

## 8.1 Solution

$$E_p = mc^2 = (1.672622 * 10^{-27} kg)(3 * 10^8 m/s)^2 = \boxed{1.5053 * 10^{-10} J}$$

$$E_p = mc^2 = (1.674927 * 10^{-27} kg)(3 * 10^8)^2 = \boxed{1.5074 * 10^{-10} J}$$

# 9 Conservation in Special Relativity

A particle of unknown mass $M$ decays into two particles of known masses $m_1 = 0.5Gev/c^2$ , and $m_2 = 1.0Gev/c^2$ , whose momenta are measured to be $\vec{p}_1 = 2.0Gev/c$ along $+y$ axis, and $\vec{p}_2 = 1.5Gev/c$ along $+x$ axis. Find the unknown mass and its velocity.

## 9.1 Solution

First of all, we know that momentum is conserved.

$$\vec{p}_i = \vec{p}_1 + \vec{p}_2 = 2\hat{j} + 1.5\hat{i} \rightarrow \vec{p}_i = \vec{p}_1 + \vec{p}_2 = 2\hat{j} + 1.5\hat{i}$$

$$\|\vec{p}_i\| = \sqrt{(2GeV/c)^2 + (1.5GeV/c)^2} = 2.5GeV/c$$

We also know that energy is conserved.

$$E_i = E_f \rightarrow \sqrt{M^2c^4 + p_i^2c^2} = \sqrt{m_1^2c^4 + p_1^2c^2} + \sqrt{m_2^2c^4 + p_2^2c^2}$$

$$(Mc^2)^2 = \left[\sqrt{m_1^2c^4 + p_1^2c^2} + \sqrt{m_2^2c^4 + p_2^2c^2}\right]^2 - (p_ic)^2$$

$$M = \frac{\sqrt{\left[\sqrt{m_1^2c^4 + p_1^2c^2} + \sqrt{m_2^2c^4 + p_2^2c^2}\right]^2 - (p_ic)^2}}{c^2}$$

Substituting $m_1 = .5GeV/c^2, m_2 = 1GeV/c^2, p_i = 2.5GeV/c, p_1 = 2Gev/c$ and $p_2 = 1.5Gev/C$, we have $\boxed{M = 0.5GeV/c^2}$. Finding the velocity of $M$ before the collision requires expanding the relativistic momentum as

$$P_M = \gamma Mu \rightarrow \frac{P_M}{M} = u\gamma \rightarrow \left(\frac{P_M}{M}\right)^2 = \frac{u^2}{1-\left(\frac{u}{c}\right)^2} \rightarrow u^2\left[1 + \left(\frac{P_M}{M_c}\right)^2\right] = \left(\frac{P_M}{M}\right)^2$$

$$u^2 = \frac{\left(\frac{P_M}{M}\right)^2}{\left[1 + \left(\frac{P_M}{M_c}\right)^2\right]} = \frac{\left(\frac{2.5GeV/c}{.5GeV/c^2}\right)^2}{\left[1 + \left(\frac{2.5GeV/c}{.5GeV/c^2}\right)^2\right]} = \boxed{0.98c}$$

# 10 Relativistic Doppler Effect

A motorist gets a ticket for running a red traffic light (wavelength 675 nm). He had some vague idea about the Doppler effect of light. Taken to court, he pleads not guilty and argues that due to the Doppler effect, the traffic light looked yellow (wavelength 575 nm) to him. The judge, well versed in relativity, accepts his plea but fines him for speeding. The judge shows leniency and lowers fine to \$0.05 (lowering from the prevailing rate of \$1.00) for each kilometer per hour that the speed exceeded the posted limit of 90 km/hour. What could be the motorist's speed? How much was the fine? Did the motorist beat the system? What is the lesson learned?

## 10.1 Solution

The Relativistic Doppler Effect states that $f' = f(1 \pm \frac{u}{c})$. We are given wavelengths here, but we know $\lambda f = c \rightarrow f = \frac{c}{\lambda}$, so that

$$f = \frac{c}{\lambda} = \frac{3 * 10^8 m/s}{675 * 10^{-9} m} = 4.44 * 10^{14} Hz, f' = \frac{c}{\lambda} = \frac{3 * 10^8 m/s}{575 * 10^{-9} m} = 5.21 * 10^{14} Hz$$

By the relativistic doppler effect, we have

$$f' = f(1 + \frac{u}{c}) \rightarrow u = c\left(\frac{f'}{f} - 1\right) = 5.217391304 * 10^7 m/s$$

The speed limit is $90km/hr = 25m/s$ and $1km/hr = 0.2777m/s$. The motorist is going $52173888.04m/s$ over the speed limit, which should incur a $\frac{52173888.04m/s}{0.2777\$/m/s} = \boxed{\$187,826,522.90}$. As the fine is in the hundreds of millions, he did not beat the system. The lesson learned is that the motorist should not have been driving at speeds sufficient to use the relativistic Doppler shift in the first place.

# 11 Relativistic Velocities

Spaceship $A$ is moving toward spaceship $B$ with a speed of $0.4c$, as measured by an astronaut in $B$. Spaceship $A$ sends a gift, $G$ toward $B$ at a speed of 0.7c relative to $A$. (a) What is the speed of $G$ relative to $B$ relativistic effects are neglected? (b) What is the speed of $G$ relative to $B$ taking relativity in consideration? (c) If the distance of $A$ from $B$ is $8 * 10^6$ km when the gift is sent out, how much time will it take the gift to reach $B$ as measured by the astronaut in $B$?

## 11.1 Solution

Should relativistic effects be neglected, we would employ a simple galilean addition of velocities to give a velocity of $0.4c + 0.7c = \boxed{1.1c}$ for $G$. Taking relativistic effects into consideration means employing the relativistic sum for velocities:

$$u_x' = \frac{u_x + v}{1 - u_x^2/c^2} = \frac{0.7c + 0.4c}{1 - (0.7c * 0.4c)/c^2} = 257812500 m/s = \boxed{2.57812500 * 10^8 m/s}$$

The time for the gift to reach $B$ is simply $\frac{d}{u_x'} = \frac{8*10^9 m}{2.57812500*10^8 m/s} = \boxed{31.03s}$